What is...the Jordan–Hölder theorem?

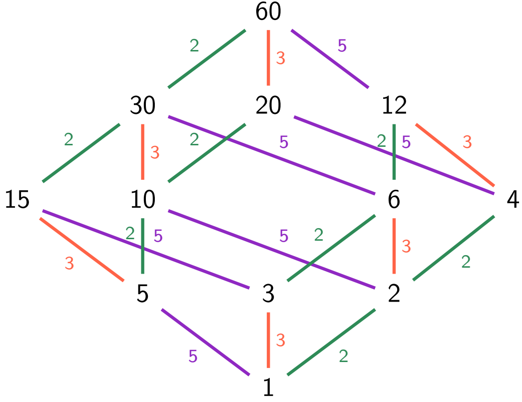
Or: Like prime numbers

The fundamental theorem of arithmetic (FTA)

Existence!

Uniqueness!

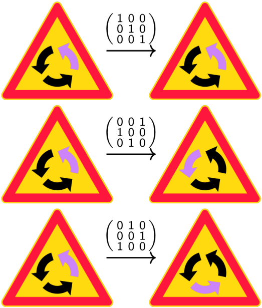
(Up to permutation)



- Prime numbers are the elements of basic multiplicative arithmetic
- These are the elements without substructure
- However, only the FTA justifies their importance

Division $\leftrightsquigarrow$ triangular block decomposition

$\mathbb{Z}/3\mathbb{Z}$ acts on



$$P = \begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & 1 \\ 1 & 0 & -1 \end{pmatrix}$$

$$P^{-1} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

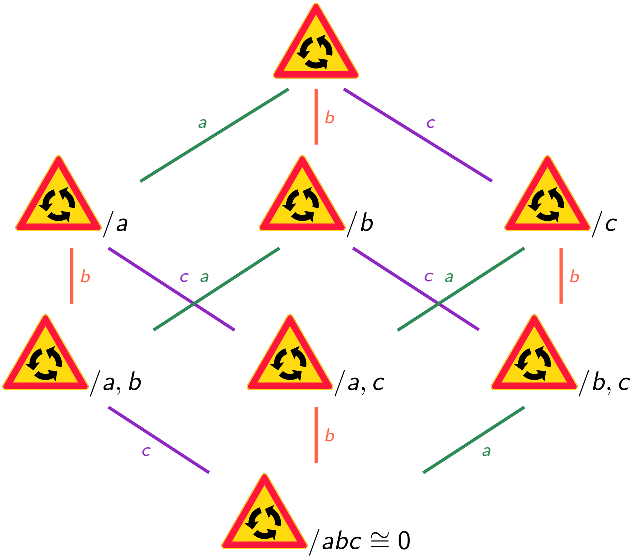
$$P^{-1} \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix}$$

$$P^{-1} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

- Simples are the elements of representation theory
- These are the elements without substructure
- Thus, we need an FTA of representation theory

A tree type picture

$$a = (1, 1, 1)$$
$$b = (e^{2\pi i/3}, e^{4\pi i/3}, 1)$$
$$c = (e^{4\pi i/3}, e^{2\pi i/3}, 1)$$



Jordan–Hölder vastly generalize the FTA

For completeness: A formal definition/statement

$\phi\colon G \to \mathrm{GL}(V)$ G-representation on a $\mathbb{K}$-vector space V

- A composition series of V is a sequence of subrepresentations

$$0 = V_0 \subset V_1 \subset \dots \subset V_{k-1} \subset V_k = V$$

  such that the factor modules V_{i+1}/V_i are simple

- k is the length of the series

Theorem We have:

- Composition series exist Existence
- If V has two series

$$0 = V_0 \subset V_1 \subset \dots \subset V_{k-1} \subset V_k = V$$
$$0 = W_0 \subset W_1 \subset \dots \subset W_{l-1} \subset W_l = V$$

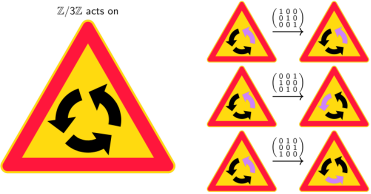
  then $k = l$ and the factors are the same up to permutation and isomorphism

$$V_{i+1}/V_i \cong W_{\sigma(i+1)}/W_{\sigma(i)}$$

  Uniqueness

Krull–Schmidt theorem

A Jordan block



- Jordan decomposition over $\mathbb{C}$ gives

$$\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix},\begin{pmatrix}0&0&1\\1&0&0\\0&1&0\end{pmatrix},\begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix}\xrightarrow[\text{change}]{\mathbb{C}\text{ base}}\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix},\begin{pmatrix}1&0&0\\0&e^{2\pi i/3}&0\\0&0&e^{4\pi i/3}\end{pmatrix},\begin{pmatrix}1&0&0\\0&e^{4\pi i/3}&0\\0&0&e^{2\pi i/3}\end{pmatrix}$$

- Jordan decomposition over $\mathbb{F}_3$ gives

$$\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix},\begin{pmatrix}0&0&1\\1&0&0\\0&1&0\end{pmatrix},\begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix}\xrightarrow[\text{change}]{\mathbb{F}_3\text{ base}}\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix},\begin{pmatrix}1&1&0\\0&1&1\\0&0&1\end{pmatrix},\begin{pmatrix}1&2&1\\0&1&2\\0&0&1\end{pmatrix}$$

There is also the analog theorem for indecomposable representations:

- Every rep is $\cong$ to a finite $\oplus$sum of indecomposable reps Existence
- Such a decomposition is unique up to permutation of summands Uniqueness

Thank you for your attention!

I hope that was of some help.