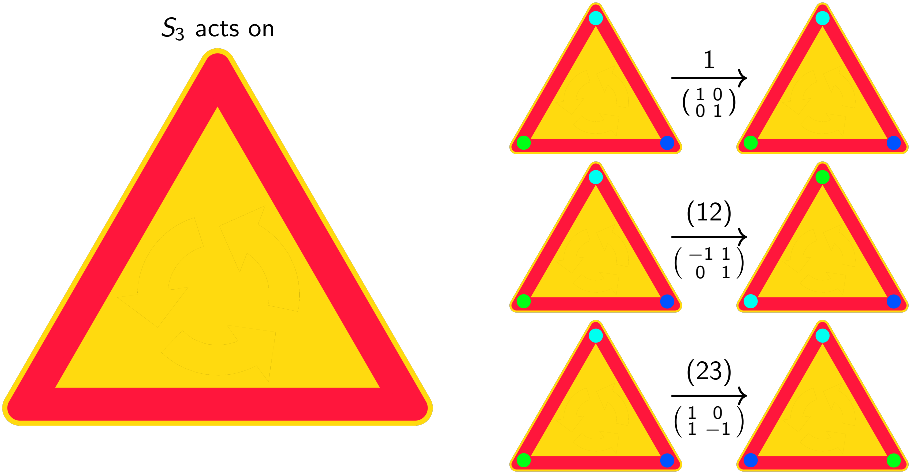
# What is...Schur's lemma?

Or: Matrices rarely commute

# The standard representation of $S_n$




- $S_n$ act on an $n-1$ simplex by permuting the vertices Permutation rep
- Getting rid of the eigenvector "sum of vertices" gives the standard rep $L_{stand}$, which is simple

## Commuting matrices

$$S = \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}$$

$$T = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}$$

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$AS - SA = \begin{pmatrix} -c & a + 2b - d \\ -2c & c \end{pmatrix}$$

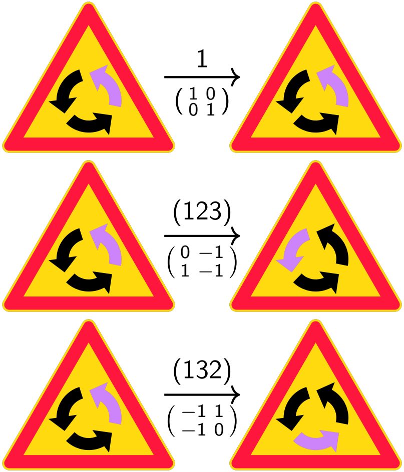
$$AT - TA = \begin{pmatrix} b & -2b \\ -a + 2c + d & -b \end{pmatrix}$$

$$AS - SA = 0 \text{ and } AT - TA = 0 \Rightarrow a = d, b = c = 0$$

- After some algebra one sees that only $(scalar \cdot id)$ commutes with $S$ and $T$
- Could this be general?

# More commuting matrices

$\mathbb{Z}/3\mathbb{Z} \subset S_3$ acts on




- The non-simple subrep of $\mathbb{Z}/3\mathbb{Z}$ above has nontrivial commuting matrices
- Precisely, $\begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$ and $\begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}$ commute

## For completeness: A formal statement

$\phi$, $\psi$ simple $G$-representation on $\mathbb{K}$-vector spaces $V$, $W$

- Any $G$-intertwiner between $\phi$ and $\psi$ is either 0 or an isomorphism
- For $\mathbb{K} = \overline{\mathbb{K}}$ any $G$-intertwiner between $\phi$ and itself is either 0 or $(\mathit{scalar} \cdot \mathit{id})$

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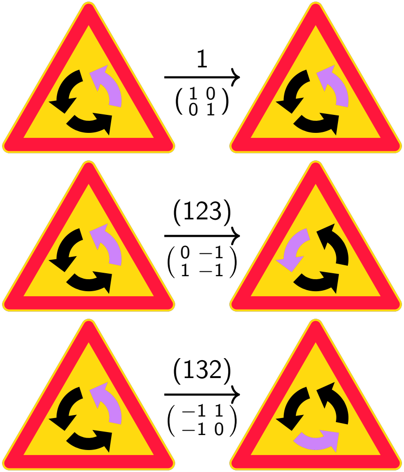
- Corollary We get that

$$\dim_{\mathbb{C}} \mathrm{Hom}_{G\text{-REP}}(V, W)$$

  is given by counting simples in $V$ and $W$ and compare overlap
- For the symmetric group $\mathbb{K} = \mathbb{Q}$ is sufficient for the second statement
- In general the condition $\mathbb{K} = \overline{\mathbb{K}}$ is necessary
  - Take for example $\mathbb{Z}/3\mathbb{Z}$
  - Over $\mathbb{Q}$ it has a 2d simple rep $V$
  - $\mathrm{End}_{\mathbb{Z}/3\mathbb{Z}\text{-REP}}(V)$ is two dimensional

## Abelian groups

$\mathbb{Z}/3\mathbb{Z} \subset S_3$ acts on




- **Corollary** (of Schur's lemma) Abelian groups have only 1d simple reps over $\mathbb{C}$
- The **converse** is also true over $\mathbb{C}$

**Thank you for your attention!**

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I hope that was of some help.