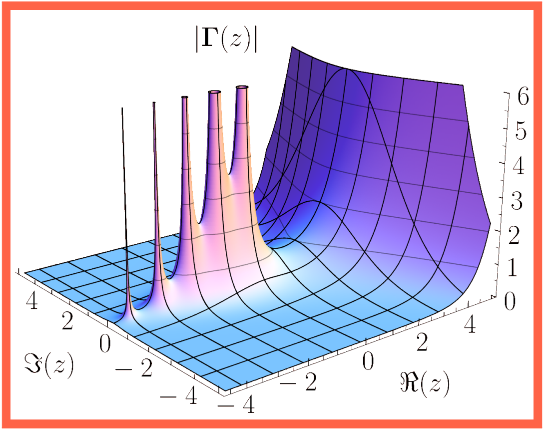
What is...tropical geometry - part 12?

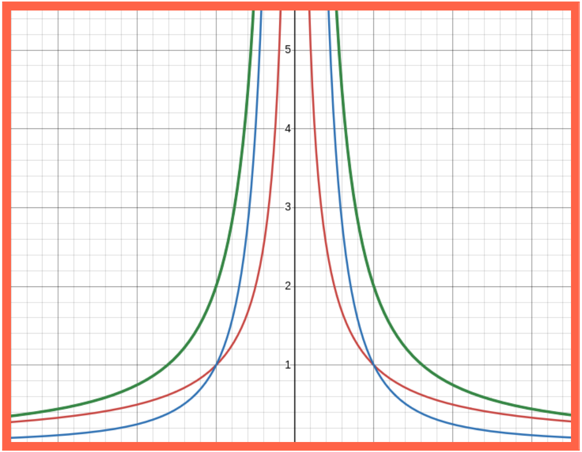
Or: Valuations

Poles and zeros



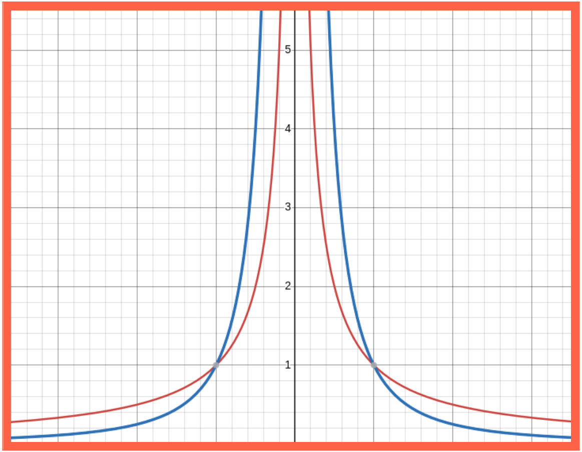
- Above The Gamma function and its poles
- Pole of f ($=$ a zero of $1/f$) $\Leftrightarrow$ dividing by zero; the order measures how often one divides by zero
- Example $\frac{1}{x^k}$ is a pole of order $-k$ at $x=0$; x^k has a zero of order k at $x=0$

Tropical addition



- Write $v(f)$ for the order of a zero or pole of f
- Addition becomes min $v(f+g) \geq \min\{v(f), v(g)\}$
- Example (above) $v(1/x^2) = -2$ and $v(1/|x| + 1/x^2) = -2$

Tropical multiplication



- Write $v(f)$ for the order of a zero or pole of f
- Multiplication becomes addition $v(fg) = v(f) + v(g)$
- Example (above) $v(1/|x|) = -1$ and $v(1/x^2) = -2 = v(1/|x|) + v(1/|x|)$

For completeness: A formal definition

A valuation on a field $\mathbb{K}$ is a function

$$v : \mathbb{K} \to \mathbb{R} \cup \{\infty\}$$

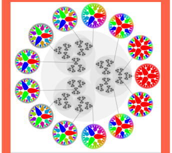
such that:

(i) $v(f) = 0$ if and only if $f = 0$

(ii) $v(fg) = v(f) + v(g)$

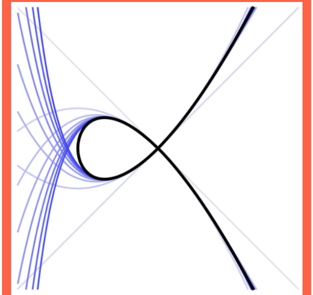
(iii) $v(f+g) \geq \min\{v(f), v(g)\}$ for $f, g \neq 0$

- Example (very tropical) $\mathbb{K}$ = rational functions, v = zero/pole order
- Example (not so tropical) The p-adic integers $\mathbb{Q}_p$ is the completion of $\mathbb{Q}$ with respect to the p-adic valuation



The 3-adic integers, with selected corresponding characters on their Pontryagin dual group

Very tropical



Truncated Puiseux expansions for the cubic curve $y^2 = x^3 + x^2$ at the double point $x = y = 0$. Darker colors indicate more terms.

- Power series $=$ something of the form $a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots$
- Puiseux series $\mathbb{C}\{\{x\}\}$ $=$ power series allowing rational numbers that have a common denominator
- $\mathbb{C}\{\{x\}\}$ has a (very tropical) valuation given by $v(f) =$ the lowest exponent that appears in the series

Thank you for your attention!

I hope that was of some help.