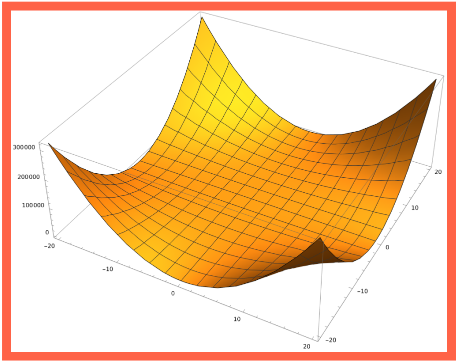
What is...tropical geometry - part 3?

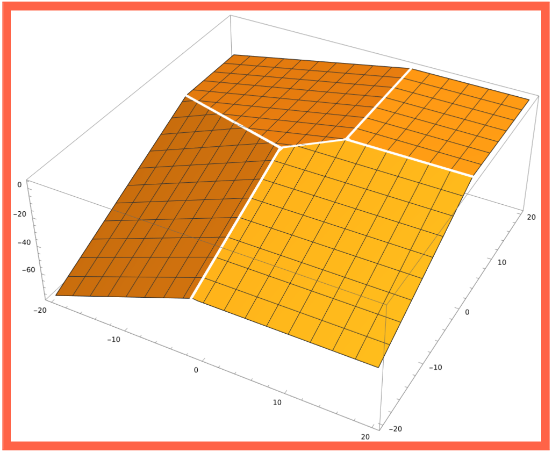
Or: Tropical polynomials

Usual polynomials



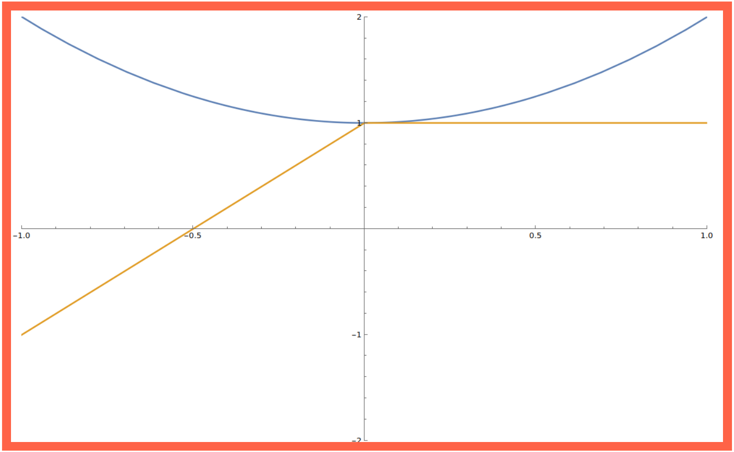
- Above The polynomial $2x^2y^2 - y^2 - x + 4$, seen as a function $\mathbb{R}^2 \to \mathbb{R}$
- Wikipedia (March 2025) A polynomial is a mathematical expression consisting of indeterminates (also called variables) and coefficients, that involves only the operations of addition, subtraction, multiplication and exponentiation to nonnegative integer powers
- Observation All of this makes sense tropically

Tropical polynomials



- Above: The tropical polynomial $2x^2y^2 - y^2 - x + 4$, seen as a function $\mathbb{R}^2 \to \mathbb{R}$
- Correct: This is $2 \otimes x \otimes x \otimes y \otimes y \oplus (-1) \otimes y \otimes y \oplus (-1) \otimes x \oplus 4$
- Translated: This is $\min\{2 + 2x + 2y, -1 + 2y, -1 + x, 4\}$

From one to the other



- Blue (top) curve above $x^2 + 1$
- Orange curve above $x^{\otimes 2} \oplus 1 = \min\{2x + 1, 1\}$
- Make it tropical $= + \mapsto \min, \circ \mapsto +$ (this maps 'poly $\mapsto$ tropical poly')

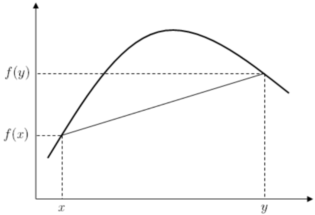
For completeness: A formal statement

A tropical polynomial is of the form

$$a \otimes x_1^{i_1} \dots x_n^{i_n} \oplus b \otimes x_1^{j_1} \dots x_n^{j_n} \oplus \dots \text{ (finitely many summands)}$$

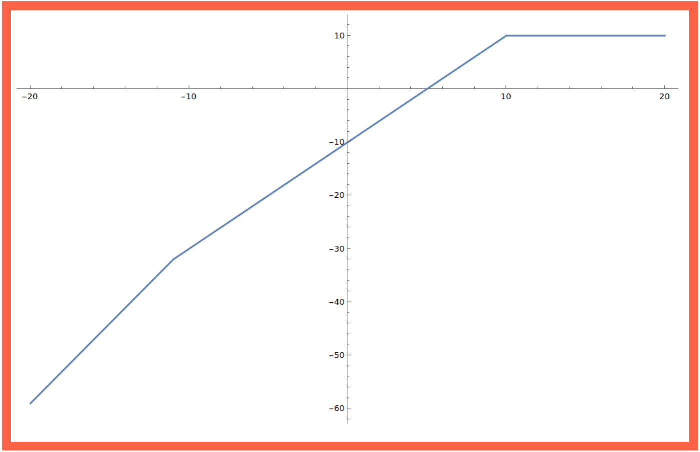
with $a, b \in \mathbb{R}$

- All the standard polynomial operations work as usual
- These are continuous, piecewise linear and concave functions $\mathbb{R}^n \to \mathbb{R}$



(Usual polynomials are continuous, but not necessarily piecewise linear or concave; crucial difference here)

Roots = breaking points



- Piecewise linear = linear up to some breakpoints
- (Tropical) roots = breakpoints of tropical polynomials
- Above $\min\{10, -10+2x, 1+3x\}$ has two breakpoints = roots

Thank you for your attention!

I hope that was of some help.