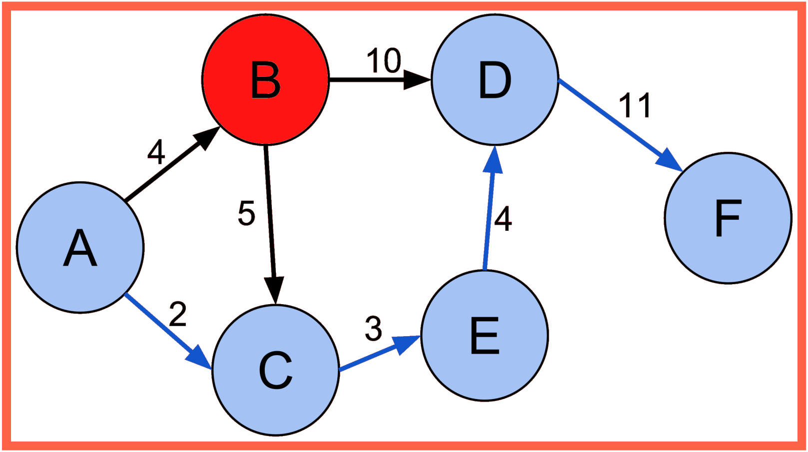
What is...tropical geometry - part 4?

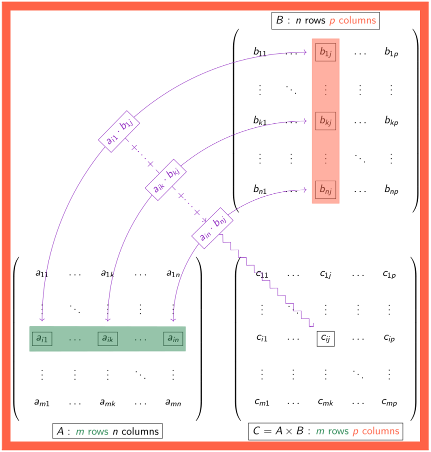
Or: Tropical matrices

Shortest path problems



- Problem Find the shortest paths in a directed weighted ("distance") graph
- Application Find directions between physical locations, e.g. driving directions
- Fun fact Solving this is problem was a motivation of the tropicals

Matrix multiplication



- Above How to multiply matrices
- Formula $c_{ij} = a_{i1}b_{1j} + \ldots + a_{in}b_{nj}$
- Observation There are only sums and multiplications

Tropical matrix multiplication

Tropical Addition Table

⊕	1	2	3	4	5	6	7
1	1	1	1	1	1	1	1
2	1	2	2	2	2	2	2
3	1	2	3	3	3	3	3
4	1	2	3	4	4	4	4
5	1	2	3	4	5	5	5
6	1	2	3	4	5	6	6
7	1	2	3	4	5	6	7

- “Tropicalization” = replace + by min and $\cdot$ by +
- Tropical formula $c_{ij} = \min\{a_{i1} + b_{1j}, \ldots, a_{in} + b_{nj}\}$
- Observation This satisfies “all” properties of usual matrix multiplication

For completeness: A formal statement

Tropical matrix addition and multiplication is exemplified by

$$\begin{pmatrix} 1 & \infty \\ 0 & -1 \end{pmatrix} \oplus \begin{pmatrix} 2 & \infty \\ \infty & -2 \end{pmatrix} = \begin{pmatrix} 1 & \infty \\ 0 & -2 \end{pmatrix}$$

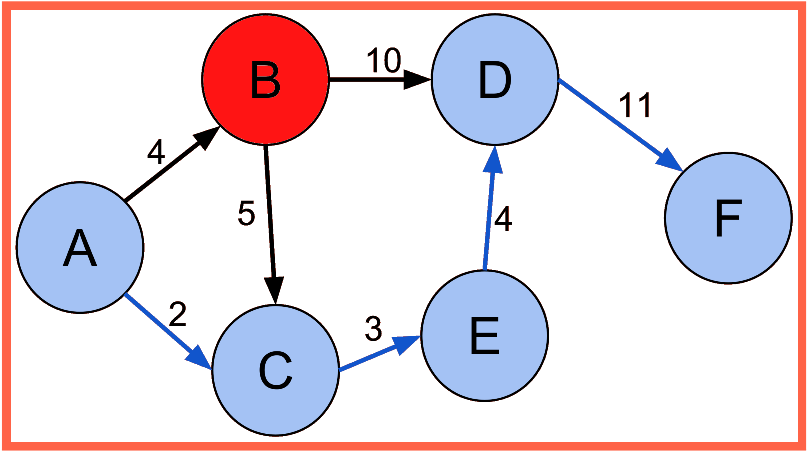
$$\begin{pmatrix} 1 & \infty \\ 0 & -1 \end{pmatrix} \otimes \begin{pmatrix} 2 & \infty \\ \infty & -2 \end{pmatrix} = \begin{pmatrix} 3 & \infty \\ 2 & -3 \end{pmatrix}$$

- All the standard operation not involving $-$ work verbatim tropically
- Complexity is $O(n^3)$ish, and that might be optimal

Year	Bound on omega	Authors
1969	2.8074	Strassen[1]
1978	2.796	Pan[9]
1979	2.780	Bini, Capovani [it], Romani[10]
1981	2.522	Schönhage[11]
1981	2.517	Romani[12]
1981	2.496	Coppersmith, Winograd[13]
1986	2.479	Strassen[14]
1990	2.3755	Coppersmith, Winograd[15]
2010	2.3737	Stothers[16]
2012	2.3729	Williams[17][18]
2014	2.3728639	Le Gall[19]
2020	2.3728596	Alman, Williams[20][21]
2022	2.371866	Duan, Wu, Zhou[22]
2024	2.371552	Williams, Xu, Xu, and Zhou[23]
2024	2.371339	Alman, Duan, Williams, Xu, Xu, and Zhou[2]

Usual matrix multiplication can be made faster; above ω for $O(n^{\omega})$

Back to shortest paths



- **Step 1** Form a tropical adjacency matrix A with ∞ if there is no edge
- **Step 2** Take the tropical power $A^{\otimes n-1}$; $n =$ number of vertices
- **Harvest** Length of shortest path from i to j is the ij entry of $A^{\otimes n-1}$

Thank you for your attention!

I hope that was of some help.