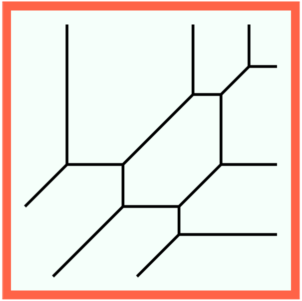
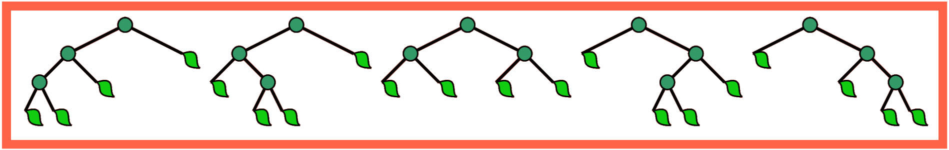
What is...tropical geometry - part 5?

Or: Linear programming

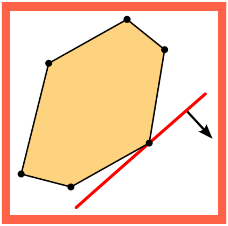
Tropical geometry = combinatorics!?

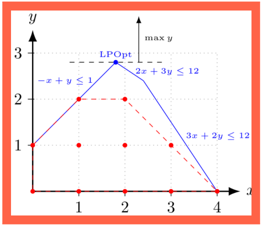




- Selling point Tropical geometry is a piecewise linear version of algebraic geometry
- Selling point 2 Tropical geometry is a combinatorial version of algebraic geometry
- Today Tropical arithmetic in linear programming

Linear programming





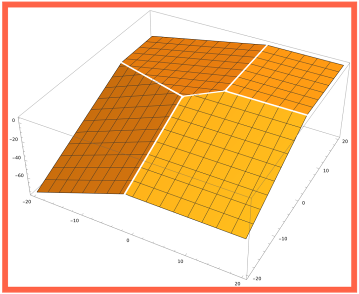
- Linear programming (LP) = find best outcome under linear requirements

Linear programs are problems that can be expressed in standard form as

Find a vector	$\mathbf{x}$
that maximizes	$\mathbf{c}^\mathsf{T}\mathbf{x}$
subject to	$A\mathbf{x} \leq \mathbf{b}$
and	$\mathbf{x} \geq \mathbf{0}.$

- Integer LP (ILP) The same, but finding integer solutions
- Fun fact IPL problems can be solved with tropical arithmetic

Tropical polynomials (reminder)



- Above: The tropical polynomial $2x^2y^2 - y^2 - x + 4$, seen as a function $\mathbb{R}^2 \to \mathbb{R}$
- Correct: This is $2 \otimes x \otimes x \otimes y \otimes y \oplus (-1) \otimes y \otimes y \oplus (-1) \otimes x \oplus 4$
- Translated: This is $\min\{2 + 2x + 2y, -1 + 2y, -1 + x, 4\}$

- A tropical polynomial is of the form

$$a \otimes x_1^{i_1} \dots x_n^{i_n} \oplus b \otimes x_1^{j_1} \dots x_n^{j_n} \oplus \dots \text{ (finitely many summands)}$$

with $a, b \in \mathbb{R}$

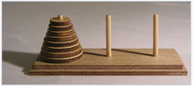
For completeness: A formal statement

Minimize $w \cdot u$ subject to $u \in \mathbb{N}^n$, $Au = b$, assuming columns of A sum to the same number a and $b_1 + \ldots + b_d = am$, so $u_1 + \cdots + u_n = m$; form:

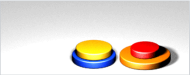
$$w_1 \odot x_1^{a_{11}} \odot \cdots \odot x_d^{a_{d1}} \oplus \cdots \oplus w_n \odot x_1^{a_{1n}} \odot \cdots \odot x_d^{a_{dn}}$$

Optimal value is the coefficient of $x_1^{b_1} \cdots x_d^{b_d}$ in the m-th power

- Tropical approach Take polynomial powers!
- This is similar to the dynamical programming approach to ILP

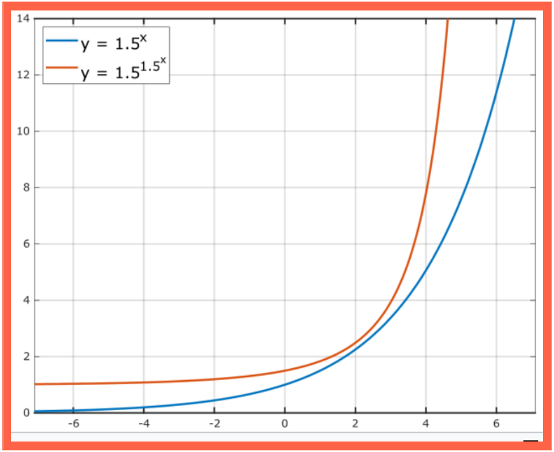


A model set of the Towers of Hanoi (with 8 disks)



An animated solution of the **Tower of Hanoi** puzzle for *T(4,3)*.

This is very difficult!



- ILP is known to be very difficult
- Example The original algorithm had runtime $\approx O(2^{n^3})$
- Often these problems are doubly exponential

Thank you for your attention!

I hope that was of some help.