

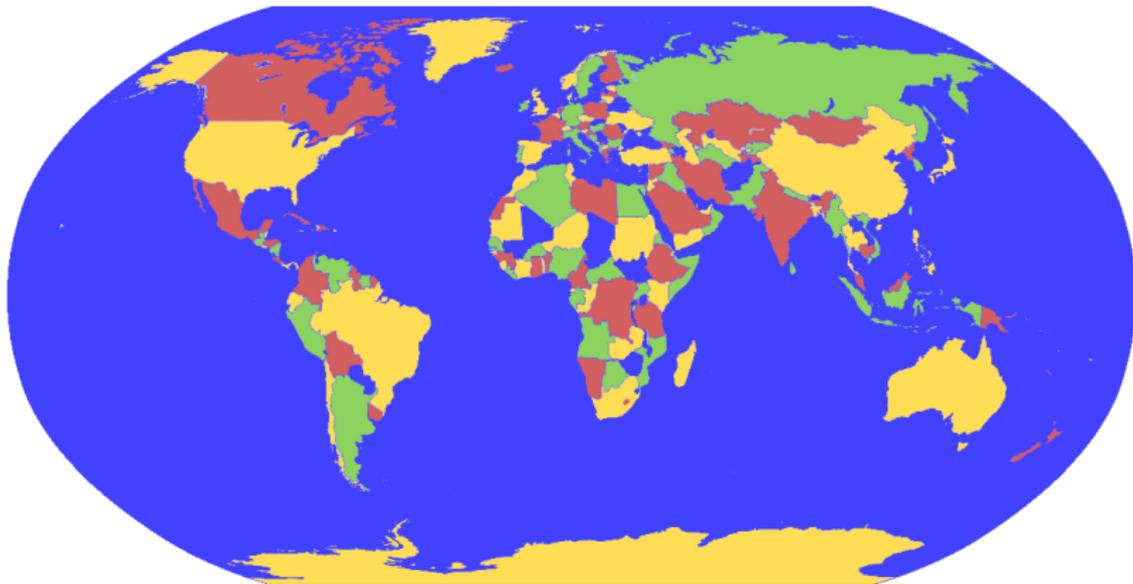
Three colors suffice

Or: SO3 webs in action

Accept **Change** what you cannot **change** accept

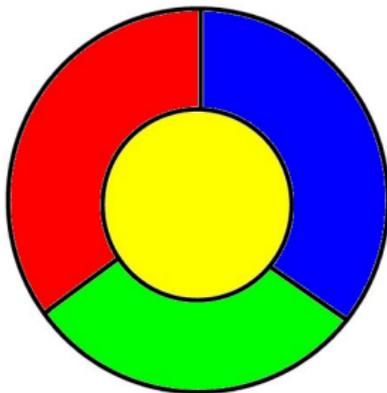
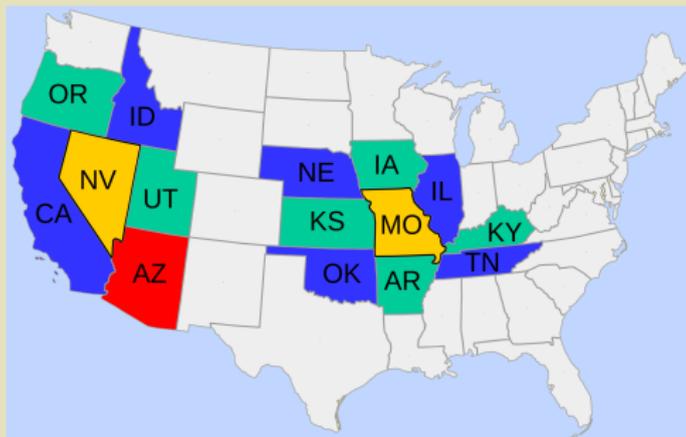


I report on work of Tait, Temperley–Lieb, Yamada & Turaev, and many more



- ▶ **Task** Color countries such that two countries that share a border get different colors
- ▶ **Above** A four coloring of the world (counting the ocean as a country)
- ▶ **Question** How many colors are needed when varying over all maps?

It is easy to see that one needs **at least** four colors:



▶ Task Color

▶ Above A f

▶ Question

et different colors

a country)

maps?

Tait's webs

My dear Thomson

A student of mine asked me to day to give him a reason for a fact which I did not know was a fact - and he has now a fact - and he has not yet. He says that if a figure to any has divided and the compartments different colored so that figures with any path of common boundary lines be differently colored - for colors may be wanted at not more - the following is his case in which four are wanted

A B C D
A B C D



Can you not see a necessity for four a more be considered for as I see at this moment, if four compartments have, and boundary line is common with one of the others, then of them either the fourth, and prevent any fourth from remaining with it. If this be true, four colors will color any figure, map without any necessity for the color meeting color beside at a point.

Has it ever been that drawing three compartments with common boundary A B C two and two - you cannot

make a fourth
then boundary from
all, making
it is tricky, with
and I, an attempt,
all conclusions - what do
you say? has he it, if
truth be as he said? the
small says he pushed it in
coloring a map of England

B is included

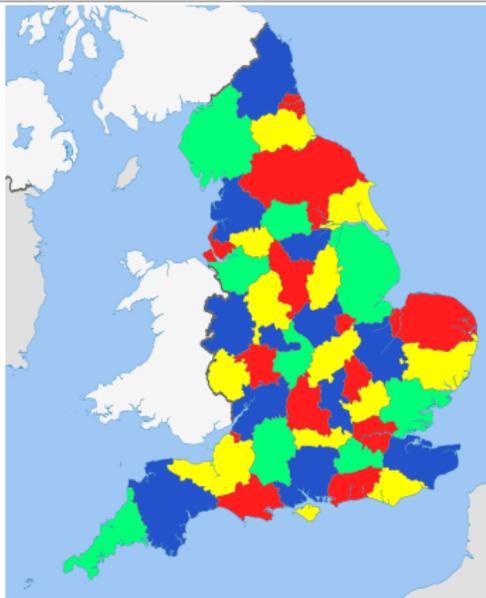


The more I think of it the more evident it seems. If you reflect with me very simple case which makes me not a student's usual, I think I must do as the theorem did. If his only be true the following proposition of logic follows

If A B C D be four names of which any two might be superseded by breaking down each with definition, then some one of the names must be a shade of some name which includes nothing essential to the other three

Yours truly
De Morgan

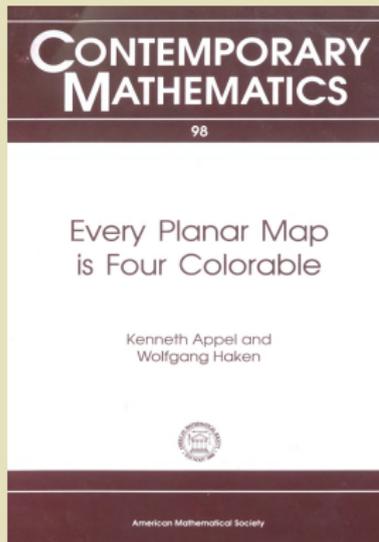
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Oct 13/61.



- ▶ Guthrie ~1852 was coloring counties of England
- ▶ They conjectured that only four colors are needed and wrote De Morgan
- ▶ De Morgan popularized the question

The 4CT (“four colors suffice”) was the first major theorem with a computer assisted proof

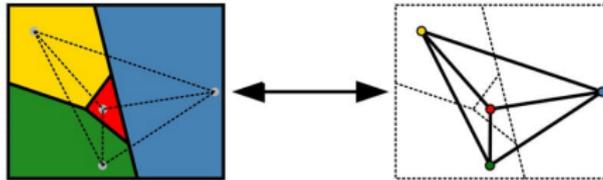
Appel–Haken’s proof ~1976 has
770ish pages
with 500 pages of
“cases to check”
The book is from ~**1989**



Today A different proof due to Tait & Temperley–Lieb & Yamada & Turaev
Spoiler The proof also has a computer component

► De Morgan popularized the question

How to go to graphs? You know the drill:



Here is an example why this is actually cool:

Northamptonshire, known as "The Rise of the Shires", is a landlocked county in the East Midlands region of England. Historically it is known for manufacturing boots and shoes. Today Northamptonshire attracts millions of visitors from around the world for its motor sports events.

The county has an area of 2,364 square kilometres or 913 square miles. As of 2020 the county boasted a population of nearly 750,000. The largest town is Kettering, followed by Wellingborough and Corby.

The county sits astride the divide that separates the basin of the Great Ouse on the east and the watershed of the Great Ouse on the west. Several important rivers in England trace their source to Northamptonshire. According to local legend "not a single book, however insignificant, flows into it from any other district".

Northamptonshire shares a common border with 8 other counties!

Coat of arms of the Marquis of Northampton

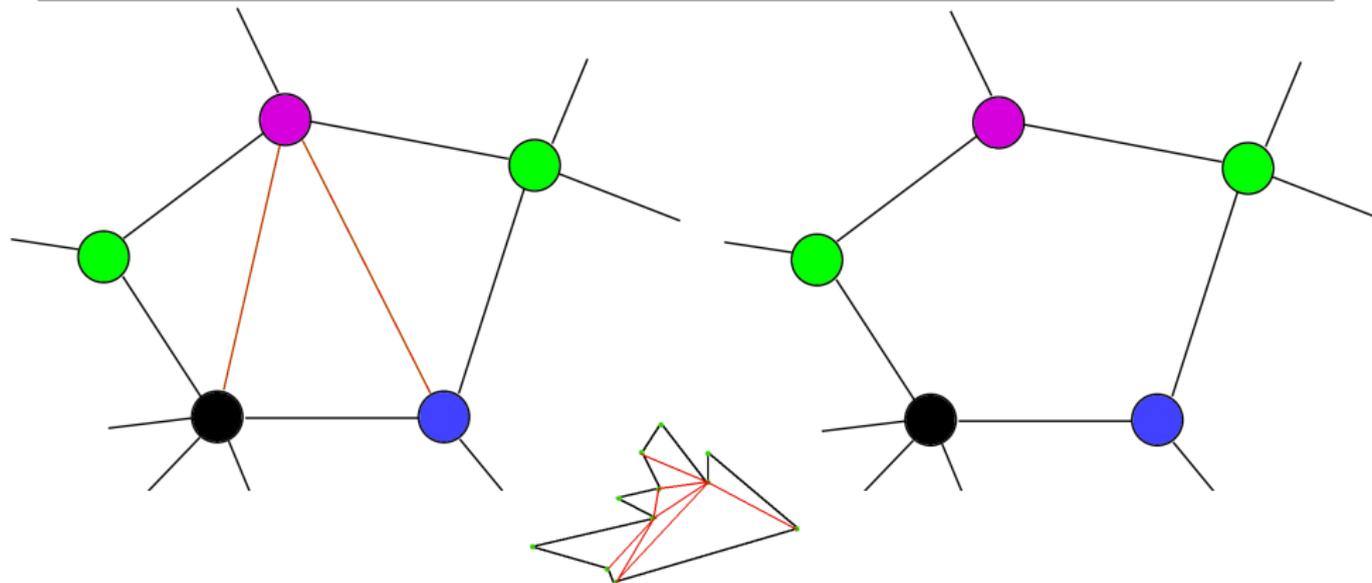
NORTHAMPTONSHIRE
Touches more adjacent ceremonial counties than any other county in the United Kingdom



Northamptonshire borders Lincolnshire with a $\approx 20m$ border

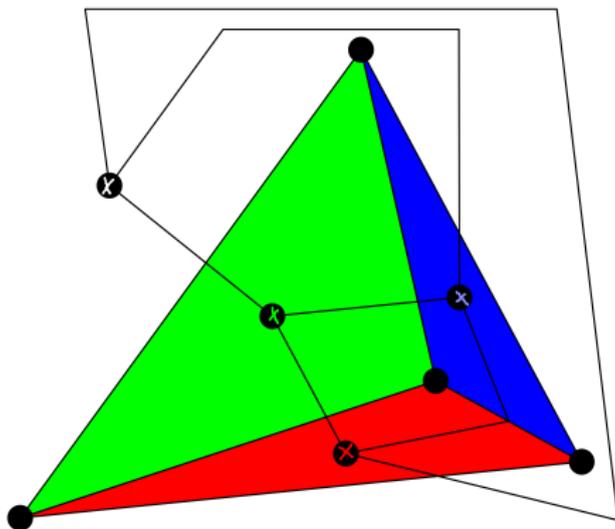
That is impossible to see on the real map!

Tait's webs



- ▶ **Math formulation** Every planar graph is four vertex-colorable
- ▶ **Tait ~1880** We can restrict to **triangulated planar graphs**
- ▶ **Why?** We can keep the coloring after removing edges!

Tait's webs



- ▶ **Tait** ~1880 We have

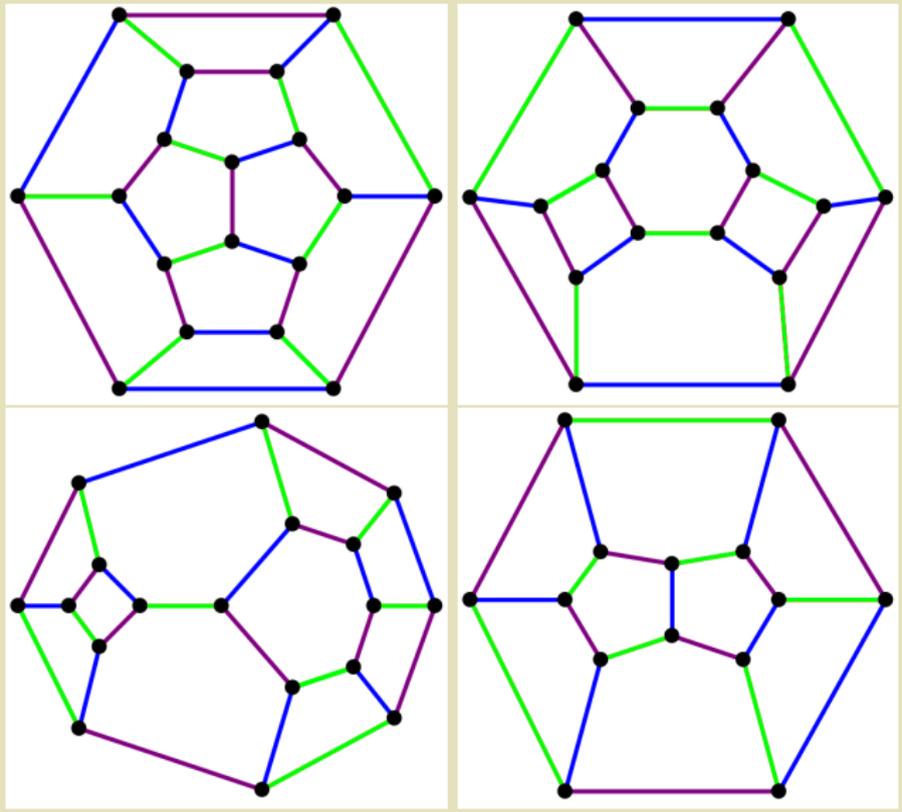
4CT (vertices) \Leftrightarrow triangulated 4CT (vertices) \Leftrightarrow trivalent 4CT (faces)

- ▶ **Why?** The dual of a triangulated planar graph is a trivalent planar graph

Tait's webs

Tait's webs = trivalent planar graphs (the name came later)

Examples



► Tait ~

4CT

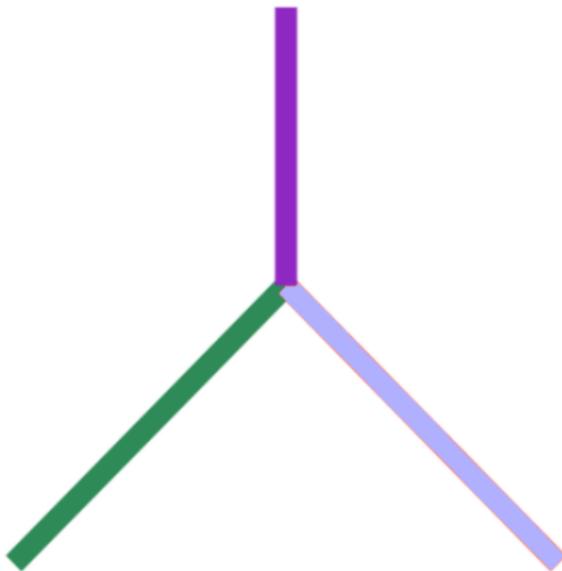
► Why?

T (faces)

ar graph

Tait's webs

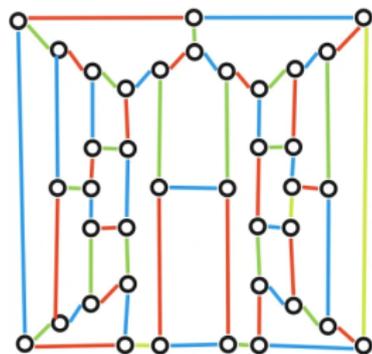
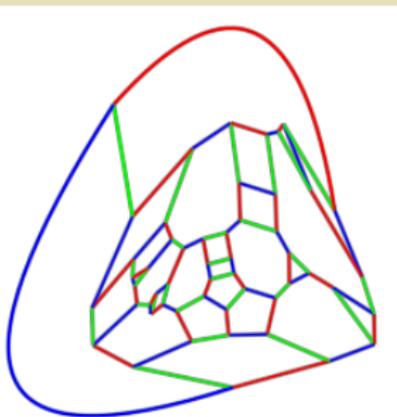
3 edge coloring :



- ▶ **3CT** is the statement that every trivalent planar bridgeless graph admits a 3 edge coloring
- ▶ **Tait** ~1880 We have

4CT (vertices) \Leftrightarrow 3CT (edges)

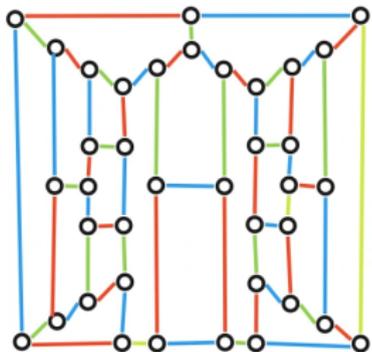
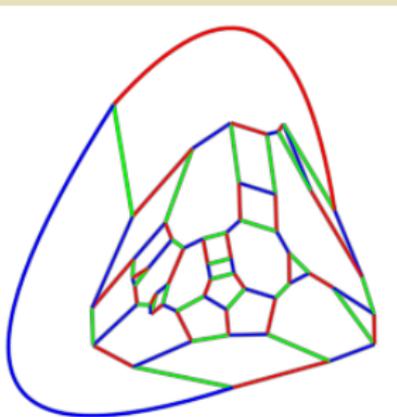
Examples



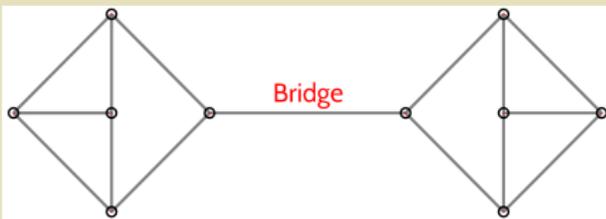
Every trivalent planar bridgeless graph admits a 3

vertices) \Leftrightarrow 3CT (edges)

Examples



Bridge = edge that disconnects
the graph after removal



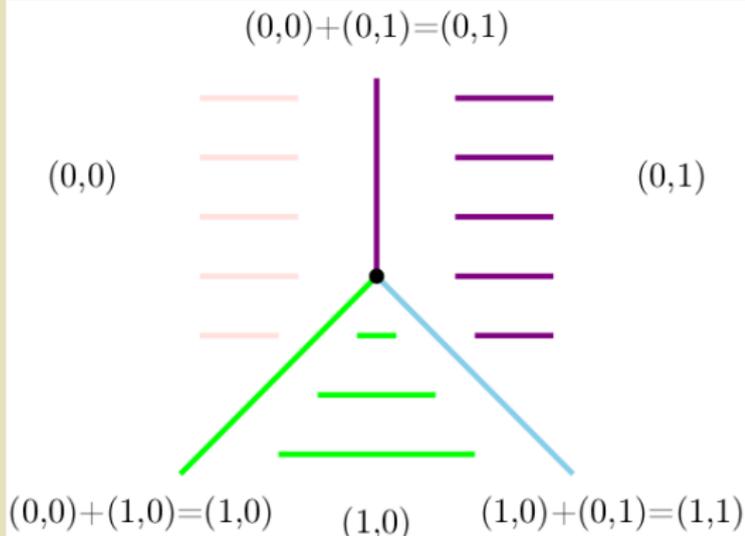
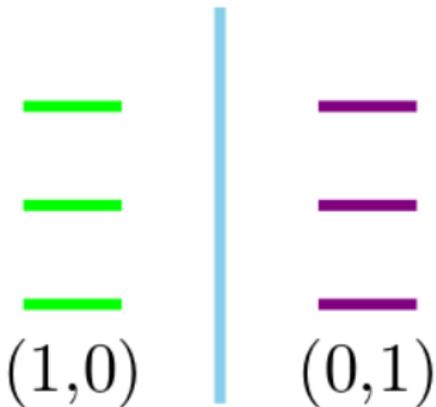
"Bridges \iff countries touching themselves"
so we disregard them

vertices) \iff 3CT (edges)

Proof sketch: 4CT \Rightarrow 3CT

Identify the 4 colors with elements of $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$
and use the rule $(a) + (b) = (a + b)$:

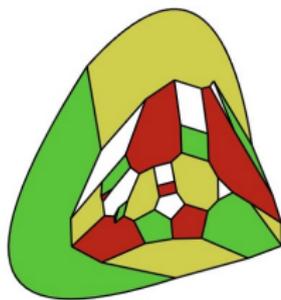
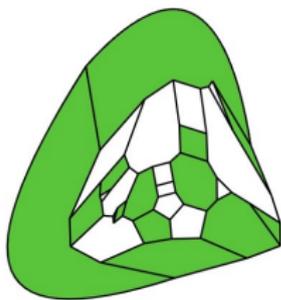
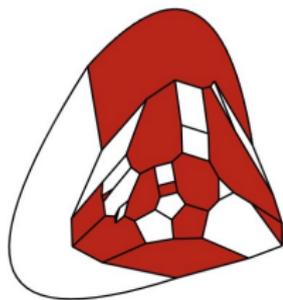
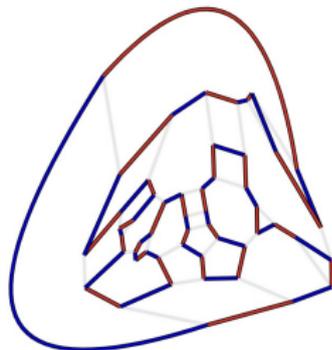
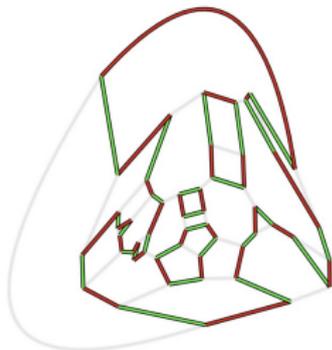
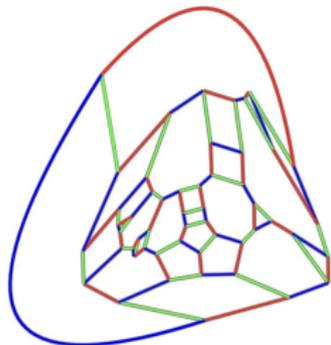
$$(1,0) + (0,1) = (1,1)$$



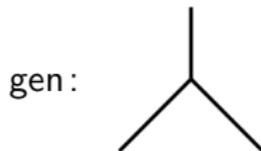
+	(0,0)	(1,0)	(0,1)	(1,1)
(0,0)	(0,0)	(1,0)	(0,1)	(1,1)
(1,0)	(1,0)	(0,0)	(1,1)	(0,1)
(0,1)	(0,1)	(1,1)	(0,0)	(1,0)
(1,1)	(1,1)	(0,1)	(1,0)	(0,0)

Proof sketch: $4CT \Leftarrow 3CT$

Identify the 4 colors with elements of $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$
get “two-color subgraphs” color and overlay them
and use the rule $(a) + (b) = (a + b)$:



Temperley–Lieb's webs



$$\bigcirc = 3, \text{ 0-gon}$$

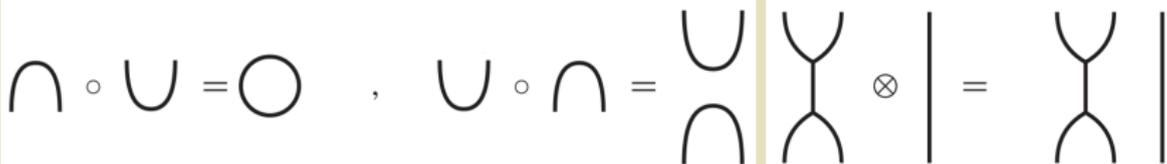
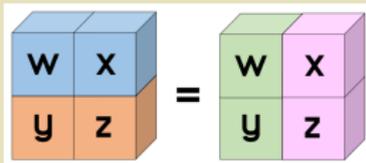
rels: = 0, 1-gon

$$\begin{array}{c} \diagdown \quad \diagup \\ \quad \quad \quad | \\ \diagup \quad \diagdown \end{array} + \begin{array}{c} \text{---} \\ \text{---} \end{array} = \begin{array}{c} \diagdown \quad \diagup \\ \text{---} \\ \diagup \quad \diagdown \end{array} + \left. \begin{array}{c} \text{---} \\ \text{---} \end{array} \right) \left(\right.$$

► Define a category **Web**(SO₃) with: **Objects** = $\bullet^{\otimes n}$; **Morphisms** = everything you can get from a trivalent vertex

► Make it **\mathbb{Z} -linear** and quotient by the relations above + isotopies

We define this as a monoidal category with
 \circ = vertical stacking, \otimes = horizontal juxtaposition

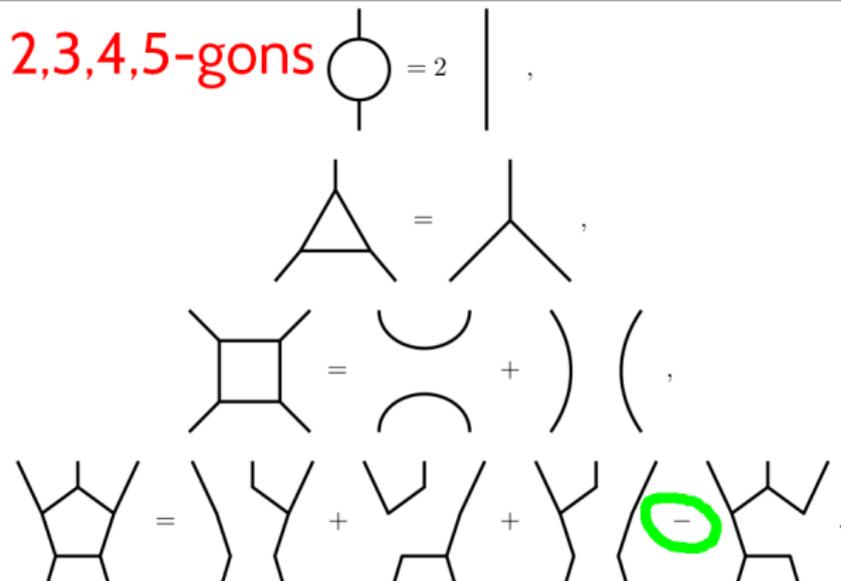


Isotopy relations are of the form



and the interesting relations are the combinatorial ones

Temperley–Lieb's webs



- ▶ The above relations **hold**
- ▶ **Upshot** Every web evaluates to a number in **Web(SO3)** **Web evaluation**
- ▶ **Essentially done by Temperley–Lieb ~1971** The web evaluation counts the number of 3-colorings

2,3,4

Why are 0-5 gon relations enough? Well:

Lemma Every web contains a ≤ 5 gon

Proof Use the Euler characteristic



This is a closed web!

Web evaluation

► The above relat

► **Upshot** Every

► **Essentially done by Temperley–Lieb** 1971 The web evaluation counts the number of 3-colorings

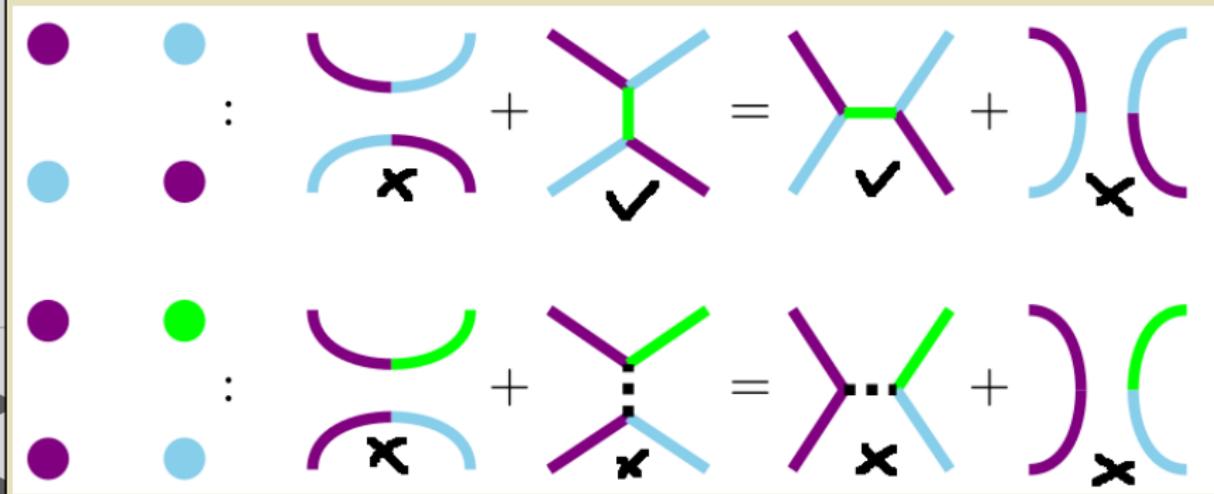
Temperley–Lieb's webs

2,3,4,5-gons  = 2

Why does the evaluation count colorings? Well:

Lemma The relations preserve the number of colors

Proof Color the boundary and check, e.g.:



- ▶ Essentially done by Temperley–Lieb ~1971 The web evaluation counts the number of 3-colorings

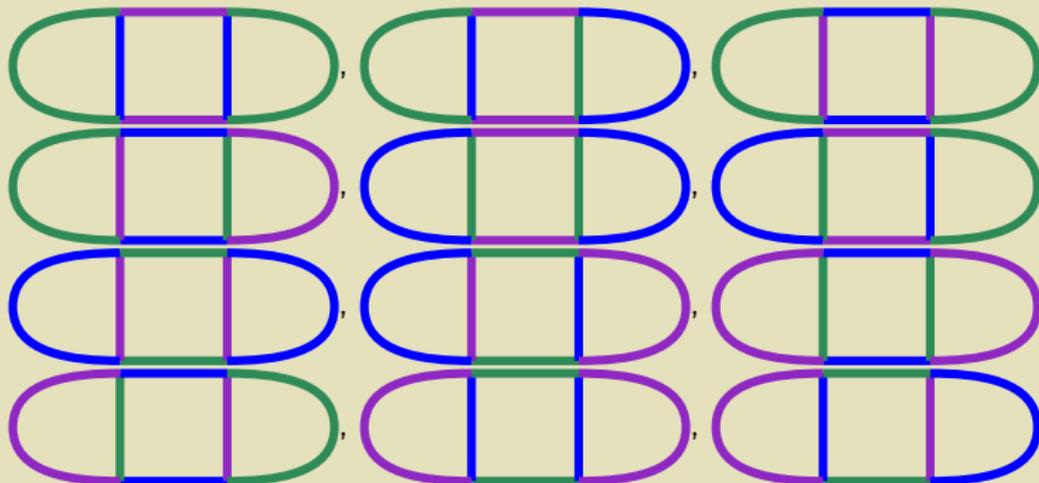
Temperley–Lieb's webs

Example



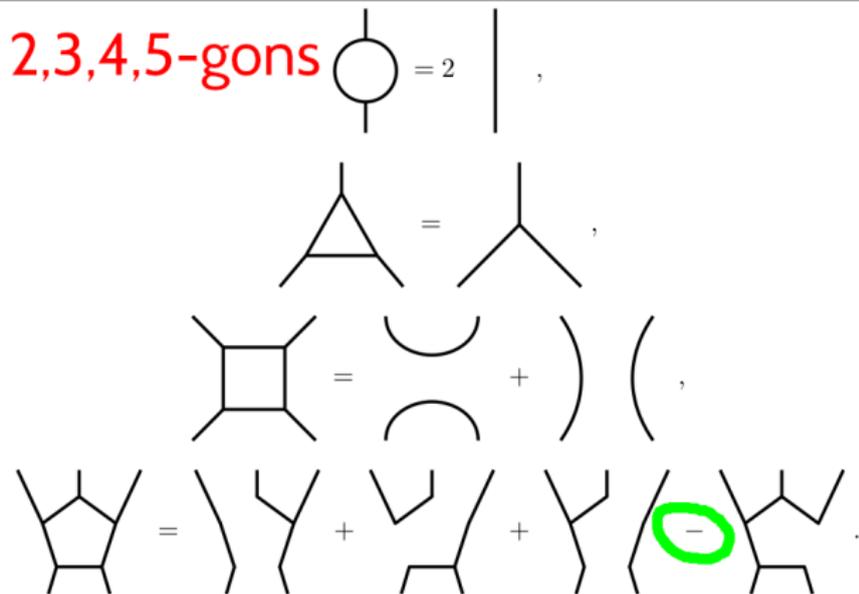
A diagram illustrating the decomposition of a web with three vertical strands into a web with one strand and a web with two strands. The equation is: $\text{web with 3 strands} = \text{web with 1 strand} + \text{web with 2 strands}$. To the right, the evaluation of the web with two strands is given as $D = 3 + 3^2 = 12$.

We have indeed twelve 3-colorings:



- ▶ Essentially done by Temperley–Lieb ~1971 The web evaluation counts the number of 3-colorings

Temperley–Lieb's webs



- ▶ **Essentially done by Temperley–Lieb ~1971** 4CT \Leftrightarrow every webs evaluates to a nonzero scalar
- ▶ The 4CT is then almost immediately true but there is a **sign** in the pentagon relation, and there **might be cancellations**

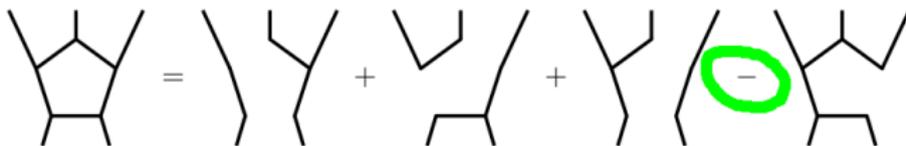
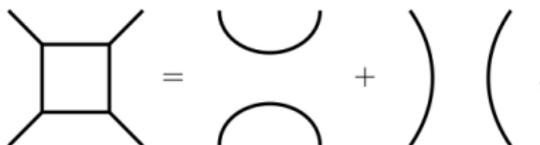
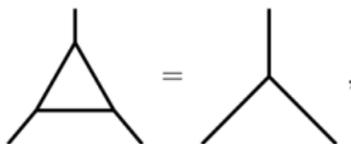
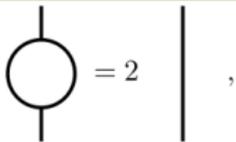
Temperley-Lieb's webs

Two questions remain:

Question 1 Can we beef this up into a proof of 4CT?

Question 2 Where do these relations come from?

2,3,4,5-gons

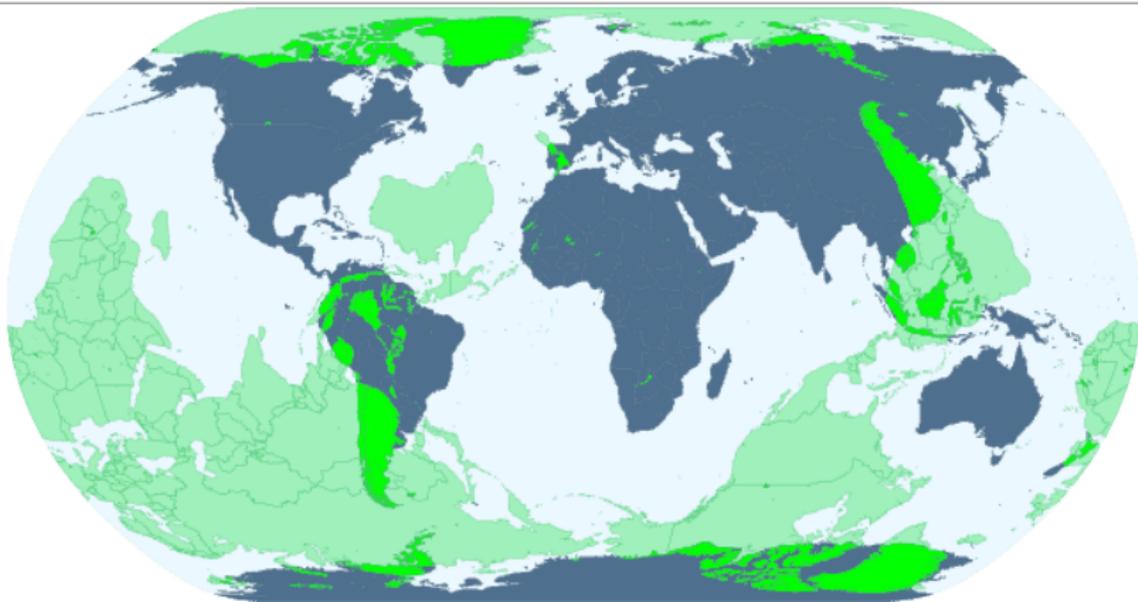


- ▶ Essen to a n
- ▶ The 4

evaluates

e pentagon

relation, and there might be cancellations



- ▶ $SO_3 = \text{rotations of } \mathbb{C}^3$
- ▶ The real version is topologically $SO_3 \cong S^2/\text{antipodal points}$
- ▶ SO_3 acts on $X = \mathbb{C}^3$ by matrices

• $\mapsto X$

 \mapsto inclusion of $\mathbb{1}$ into $X \otimes X$

 \mapsto inclusion of X into $X \otimes X$

cap =  : $X \otimes X \rightarrow \mathbb{1}$,

cup =  : $\mathbb{1} \rightarrow X \otimes X$,

tup =  : $X \otimes X \otimes X \rightarrow \mathbb{1}$,

tdown =  : $\mathbb{1} \rightarrow X \otimes X \otimes X$

- ▶ **Essentially done by Yamada & Turaev ~1989** We have

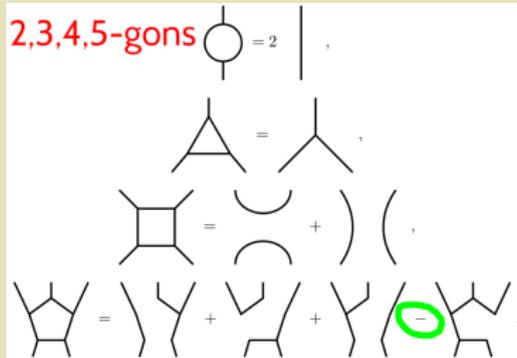
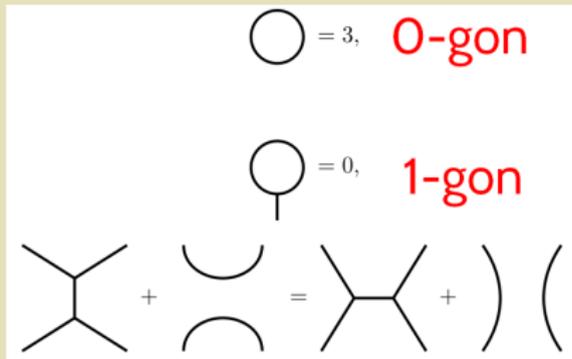
Web(SO3) \cong Rep(SO3) using the above

- ▶ **Equivalence of categories** = they encode the same information

- ▶ **Rep(SO3)** = fd reps of SO3 over \mathbb{C}

Upshot

Under $\mathbf{Web}(SO_3) \cong \mathbf{Rep}(SO_3)$ the relations, e.g.



X,
X ⊗ X

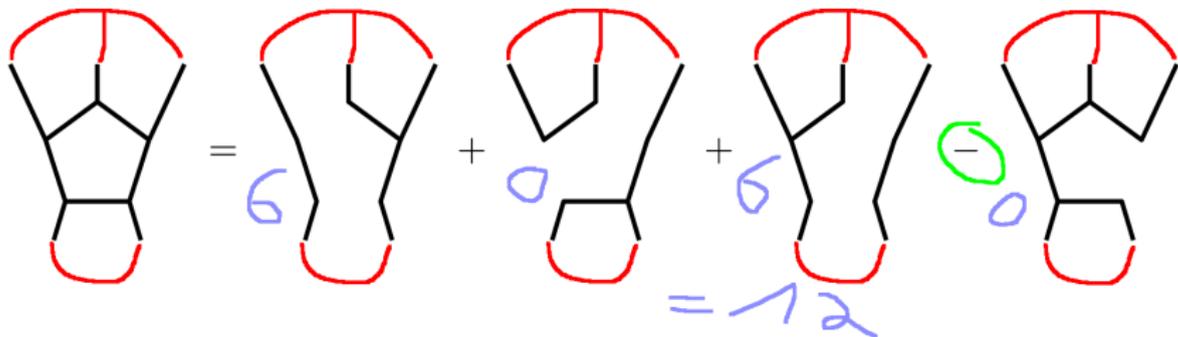
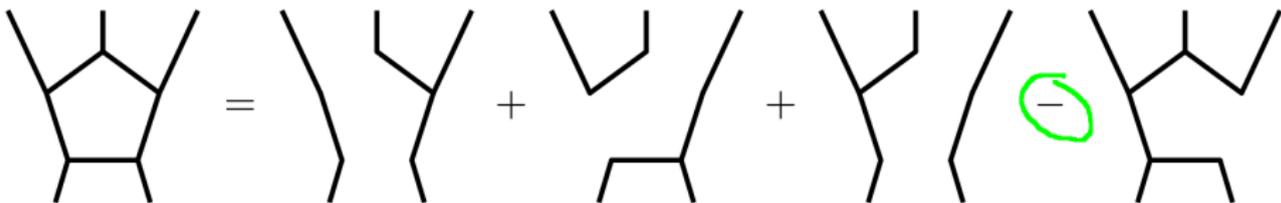
are equations between SO_3 -equivariant matrices

Example

The “I=H” relation is an equation of four 9-by-9 matrices

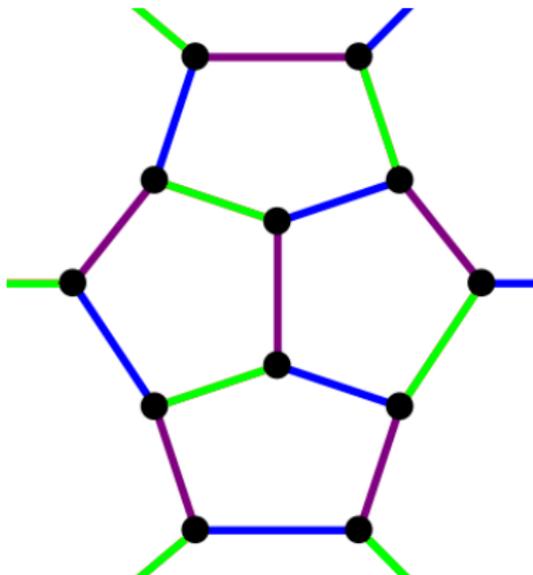
- ▶ Essence of Web
- ▶ Equations
- ▶ Relations

Yamada & Turaev's webs



- ▶ Recall that we want to show **positivity** but the pentagon has a sign
- ▶ **Observation** Closing the pentagon “minimally” evaluates to a positive number

An unavoidable configuration:



- ▶ An Euler characteristic argument shows that there are finitely many “unavoidable configurations” one needs to check similarly
- ▶ There are finitely many “minimally closures” for each one of them

Theorem

(A) 4CT holds \Leftrightarrow (B) finitely many configurations evaluate to a positive number

One can computer verify that (B) holds (too many to do by hand)



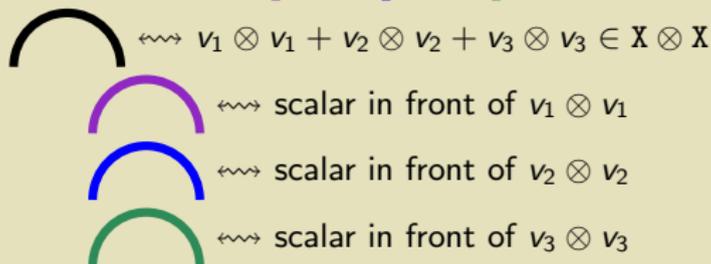
► There are finitely many "minimally closures" for each one of them

Khovanov–Kuperberg ~1999 essentially showed that 3-colorings correspond to a base-change matrix between a standard basis and a dual canonical basis in $\mathbf{Rep}(\mathbf{SO}_3)$

Example

$\{v_1, v_2, v_3\}$ basis of X

Identify: $\color{purple}{|} = v_1, \color{blue}{|} = v_2, \color{green}{|} = v_3$



Theorem

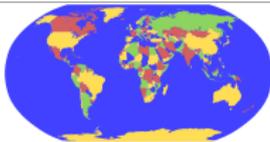
(A) 4CT holds \iff (C) finitely many inequalities of matrix entries

One can computer verify that (C) holds (too many to do by hand)

► An “una

► There are finitely many “minimally closures” for each one of them

Tait's webs



- **Tait** Color countries such that two countries that share a border get different colors
- **Above** A four coloring of the world (counting the oceans as a country)
- **Question:** How many colors are needed when varying over all maps?

Three colors suffice. (© 1993 web3 in action, January 1994, p. 1, 2.)

Tait's webs



- **SCT** is the statement that every trivalent planar bridgeless graph admits a 3 edge coloring
- **Tait - 1880** We have

4CT (vertices) \Leftrightarrow SCT (edges)

Three colors suffice. (© 1993 web3 in action, January 1994, p. 1, 2.)

Temperley-Lieb's webs

2,3,4-gons

Why are 0-5 gon relations enough? Web

Lemma: Every web contains a ≤ 5 gon

Proof: Use the Euler characteristic

► The above relation holds

► **Upshot:** Every

► **Essentially done by** Temperley-Lieb - 1971 The web evaluation counts the number of 3-colorings

This is a closed web. Web evaluation

Three colors suffice. (© 1993 web3 in action, January 1994, p. 1, 2.)

Tait's webs



- **Guthrie - 1852** was coloring counties of England
- They conjectured that **only four colors** are needed and wrote De Morgan
- De Morgan **popularized** the question

Three colors suffice. (© 1993 web3 in action, January 1994, p. 1, 2.)

Proof sketch: 4CT \Leftrightarrow SCT

Identify the 4 colors with elements of $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ and use the rule $(x) + (y) = (x + y)$

$(1,0) + (0,1) = (1,1)$

$(0,0) + (0,1) = (0,1)$

$(0,0) + (1,0) = (1,0)$

$(1,0) + (0,1) = (1,1)$

$(1,1) + (1,0) = (0,1)$

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Three colors suffice. (© 1993 web3 in action, January 1994, p. 1, 2.)

Temperley-Lieb's webs

2,3,4,5-gons

Why does the evaluation count colorings? Web

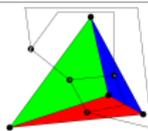
Lemma: The relations preserve the number of colors

Proof: Color the boundary and check, e.g.:

► **Essentially done by** Temperley-Lieb - 1971 The web evaluation counts the number of 3-colorings

Three colors suffice. (© 1993 web3 in action, January 1994, p. 1, 2.)

Tait's webs



- **Tait - 1880** We have

4CT (vertices) \Leftrightarrow triangulated 4CT (vertices) \Leftrightarrow trivalent 4CT (faces)

- **Why?** The dual of a triangulated planar graph is a trivalent planar graph

Three colors suffice. (© 1993 web3 in action, January 1994, p. 1, 2.)

Temperley-Lieb's webs

2,3,4,5-gons

► The above relation holds

► **Upshot:** Every web evaluates to a number in **Web(SO3)** Web evaluation

► **Essentially done by** Temperley-Lieb - 1971 The web evaluation counts the number of 3-colorings

Three colors suffice. (© 1993 web3 in action, January 1994, p. 1, 2.)

Yamada & Turanova's webs

► Recall that we want to show **positivity** but the pentagon has a sign

► **Observation:** Closing the pentagon "naturally" evaluates to a positive number

Three colors suffice. (© 1993 web3 in action, January 1994, p. 1, 2.)

Thanks for your attention!