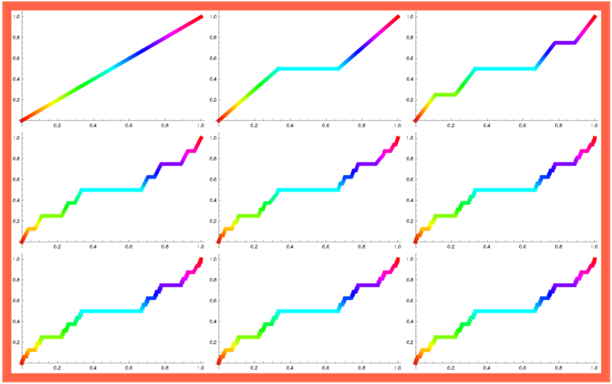
Fractal behavior in monoidal categories

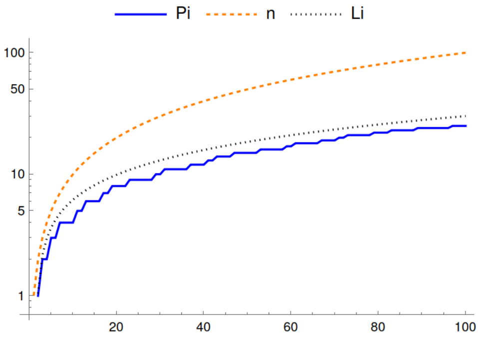
Or: SL2, Cantor and Sierpinski

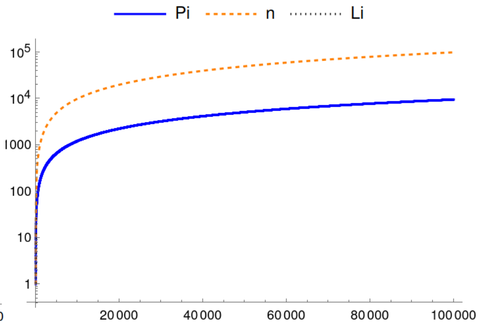
Happy pride month!



I report on work of Coulembier, Etingof and Ostrik

Let us not count!





- Prime number function $\pi(n)$ $= \#$ primes $\leq$ n
- Counting primes is very tricky as primes "pop up randomly"
- Question 1 What is the leading growth (of the number of primes)?
- Answer 1 There are roughly $c(n) \cdot n$ for sublinear correction term $c(n)$

Let us not count!

Seriously, counting is difficult!

Legendre ∼1808:
(for $n/(\ln n - 1.08366)$)

Limite x	Nombre y par la formule.	Nombre y par les Tables.	Limite x	Nombre y par la formule.	Nombre y par les Tables.
10000	1230	1230	100000	9588	9592
20000	2268	2263	150000	13844	13849
30000	3252	3246	200000	17982	17984
40000	4205	4204	250000	22035	22045
50000	5136	5134	300000	26023	25998
60000	6049	6058	350000	29961	29977
70000	6949	6936	400000	33854	33861
80000	7838	7837			
90000	8717	8713			

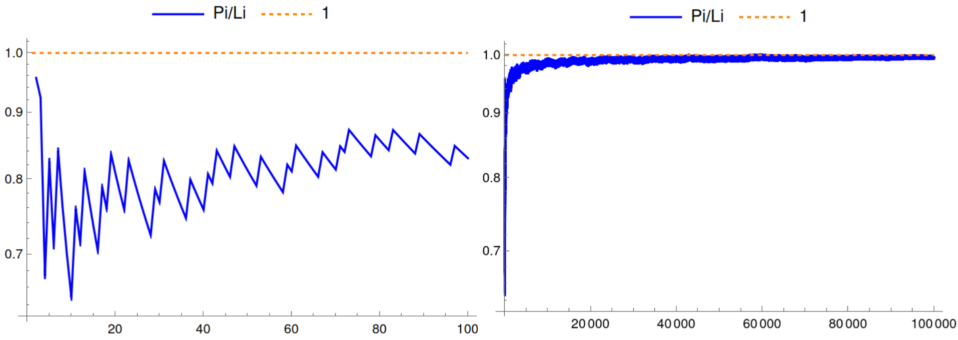
Acctually, #primes<1000 =1229...

Gauss, Legendre and company counted primes up to $n = 400000$ and more

That took years (your IPhone can do that in seconds...humans have advanced!)

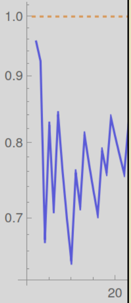
- Question 1 What is the leading growth (of the number of primes)?
- Answer 1 There are roughly $c(n) \cdot n$ for sublinear correction term $c(n)$

Let us not count!



- Asymptotically equal $f \sim g$ if $\lim_{n\to\infty} f(n)/g(n) \to 1$
- Logarithmic integral $Li(x) = \int_2^x 1/\ln(t)dt$
- Question 2 What is the growth (of the number of primes) asymptotically?
- Answer 2 We have $\pi(n) \sim n/\log(n) \sim Li(n)$

Let us n



How good is this not counting?

Riemann ∼1859 calculates “the variance”:

VII.

Ueber die Anzahl der Primzahlen unter einer gegebenen Grösse.

(Monatsberichte der Berliner Akademie, November 1859.)

Durch Einsetzung dieser Werthe in den Ausdruck von $f(x)$ erhält man

$$f(x) = Li(x) - \Sigma^{\alpha}\left(Li\left(x^{\frac{1}{2}+\alpha i}\right) + Li\left(x^{\frac{1}{2}-\alpha i}\right)\right)$$

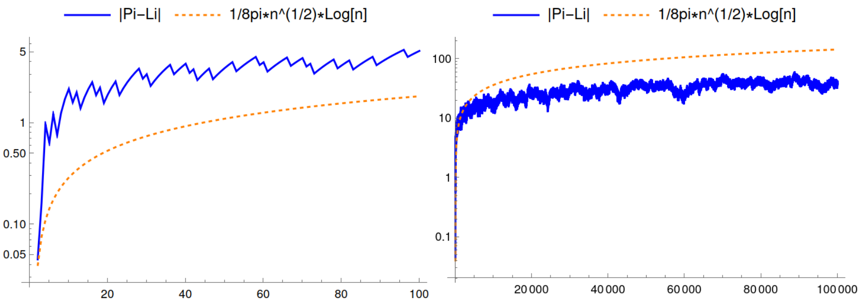
$$+ \int_x^\infty \frac{1}{x^2-1}\,\frac{dx}{x \log x} + \log \xi(0),$$

wenn in Σ^{α} für α sämmtliche positiven (oder einen positiven reellen Theil enthaltenden) Wurzeln der Gleichung $\xi(\alpha) = 0$, ihrer Grösse nach geordnet, gesetzt werden. Es lässt sich, mit Hülfe einer genaueren Discussion der Function ξ, leicht zeigen, dass bei dieser Anordnung der Werth der Reihe

f is essentially the prime counting function π

- Asy
- Log
- Que ... tically?
- Ans

Let us not count!



- Asymptotically equal does not imply that the difference is good
- $|f(n) - g(n)|$ is a measurement of how good the approximation is
- Question 3 What is variance from the expected value $(Li(n))$?
- Conjectural answer 3 We have $|\pi(n) - Li(n)| \in O(n^{1/2} \log n)$ or $|\pi(n) - Li(n)| \leq \frac{1}{8\pi} n^{1/2} \log n$ (for $n \geq 2657$)

Let us not count!

$$b_n \sim h(n) \cdot n^{\tau} \cdot \beta^n$$

$h\colon \mathbb{Z}_{\geq 0} \to \mathbb{R}_{>0}$ is a function *bounded away from* $0, \infty$,
n^{τ} is the *subexponential factor*, $\tau \in \mathbb{R}$,
β^n is the *exponential factor*, $\beta \in \mathbb{R}_{\geq 1}$.

- Ansatz: What to expect from not counting
- Any sequence of numbers b_n counting something (in monoidal categories) often satisfies the above
- h is often a constant but sometimes h is more complicated

Let us not count!

Examples

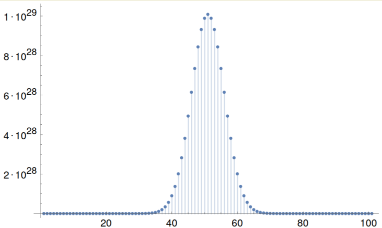
(1) (Prime counting function $\pi'(n) = \pi(e^n)$) $\sim 1 \cdot n^{-1} \cdot e^n$

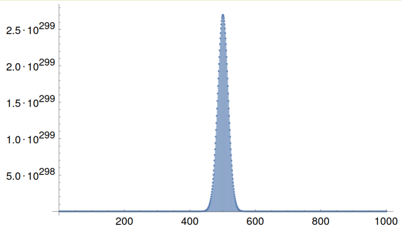
(2) (Number of partitions of n^2) $\sim 1/(4\sqrt{3}) \cdot n^{-2} \cdot (e^{\sqrt{2/3}\pi})^n$

(3) (Number of trees with n vertices) $\sim C \cdot n^{-5/2} \cdot D^n$ with $C \approx 0.535$, $D \approx 2.996$

(4) (Rabbit counting à la Fibonacci) $\sim \sqrt{5} \cdot 1 = n^0 \cdot \phi^n$

(5) (Middle binomials $\binom{n}{n/2}$) $\sim \sqrt{2/\pi} \cdot n^{-1/2} \cdot 2^n$





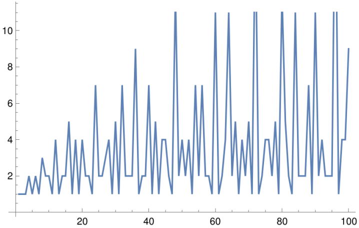
often satisfies the above

- h is often a constant but sometimes h is more complicated

Let us not count!

Examples (cont.)

#mult. partitions:



- **Multiplicative partition** = an ordered way of writing an integer as a product of positive integers ≥ 2, e.g. $12 = 3 \cdot 2 \cdot 2$

Better to look at: the average $\sum_{k=1}^{n} m(k)/n$, $m(k)$=#mult. partitions

(Average #mult. partitions of e^{n^2}) $\sim 1/(2\pi) \cdot n^{-3/2} \cdot (e^2)^n$

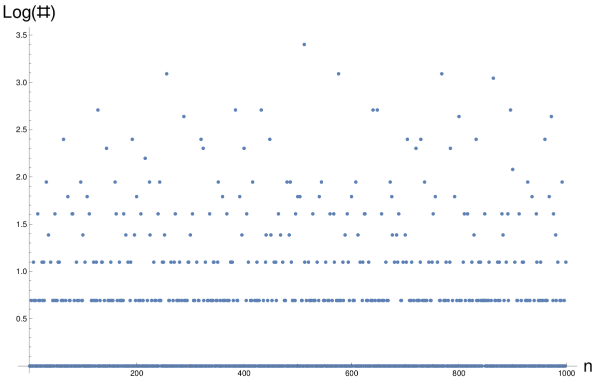
often satisfies the above

- h is often a constant but sometimes h is more complicated

Examples (cont.)

A000688 Number of Abelian groups of order n; number of factorizations of n into prime powers. 129
(Formerly M0064 N0020)

1, 1, 1, 2, 1, 1, 1, 3, 2, 1, 1, 2, 1, 1, 1, 5, 1, 2, 1, 2, 1, 1, 1, 3, 2, 1, 3, 2, 1, 1, 1, 7, 1, 1, 1, 4, 1, 1, 1, 3, 1, 1, 1, 2, 2, 1, 1, 5, 2, 2, 1, 2, 1, 3, 1, 3, 1, 1, 1, 2, 1, 1, 2, 11, 1, 1, 1, 2, 1, 1, 1, 6, 1, 1, 2, 2, 1, 1, 1, 5, 5, 1, 1, 2, 1, 1, 1, 3, 1, 2, 1, 2, 1, 1, 1, 7, 1, 2, 2, 4, 1, 1, 1, 3, 1, 1, 1 (list; graph; refs; listen; history; text; internal format)



The average $\sum_{k=1}^{n} agnu(k)/n$, $agnu(k)$=#abelian groups of order k

(Average #abelian groups of order n) $\sim$ $\prod_{j\geq 2}\zeta(j)$ $\cdot$ $1 = n^0$ $\cdot$ $1 = 1^n$, $\prod_{j\geq 2}\zeta(j) \approx 2.295$

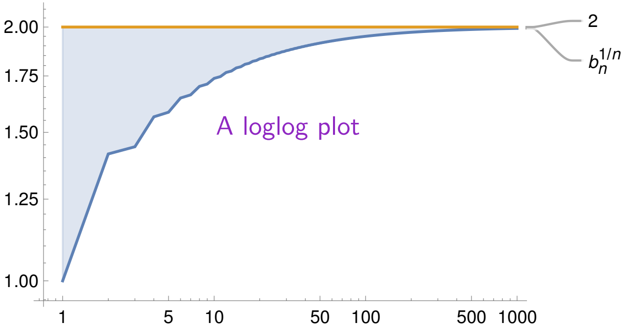
Let us not count!

- Task For various counts b_n in monoidal categories where counting is too hard try to find:
  - The dominating growth β
  - An asymptotic formula "$a(n) = h \cdot n^{\tau} \cdot \beta^{n}$"
  - If possible bound the variance $|b_n - a_n|$

⊗ cat counts – char zero

- Γ = a group-thing (more details later)
- $\mathbb{K}$ = any ground field, V = any fin dim Γ-rep
- Problem Decompose $V^{\otimes n}$ - too difficult, better: count summands

$\otimes$ cat counts – char zero

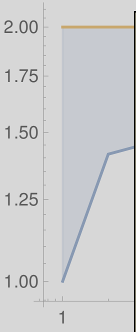


- $b_n = b_n^{\Gamma,V}$=number of indecomposable summands of $V^{\otimes n}$ (with multiplicities)
- **Example** $\Gamma = SL_2$, $\mathbb{K} = \mathbb{C}$, $V = \mathbb{C}^2$ (vector rep), then

$$\{1, 1, 2, 3, 6, 10, 20, 35, 70, 126, 252\}, \quad b_n \text{ for } n = 0, ..., 10.$$

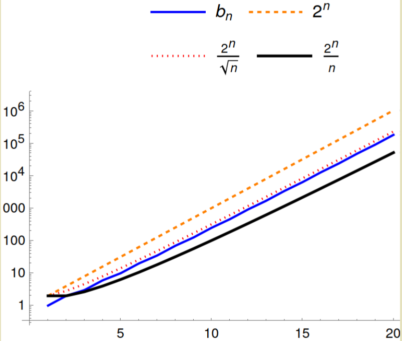
Research task Copy the sequence and put it into OEIS

⊗ cat counts – char zero



b_n = middle binomials Hence:

$$b_n \sim \sqrt{2/\pi} \cdot n^{-1/2} \cdot 2^n$$



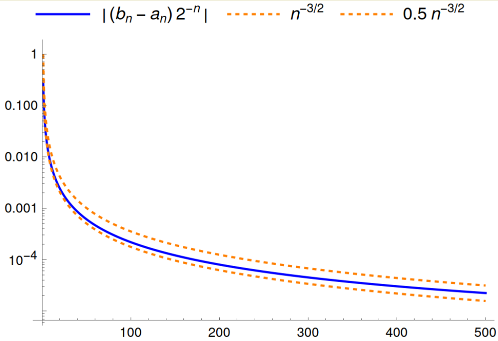
- $b_n = b_n^{\Gamma,V}$=numbe ... $^{\otimes n}$ (with multiplicities)
- **Example** $\Gamma = SL_2$ ...

$$\{1, 1, 2, 3, 6, 10, 20, 35, 70, 126, 252\}, \quad b_n \text{ for } n = 0, ..., 10.$$

Research task Copy the sequence and put it into OEIS

Even the variance is doable :

$$|b_n - a_n| \sim C \cdot n^{-3/2} \cdot 2^n$$



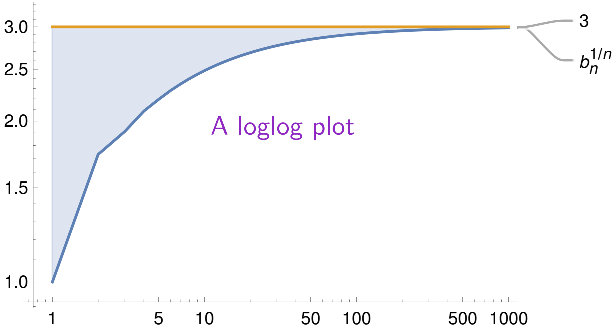
Proof? Scroll through the OEIS page for the middle binomials

- $b_n = b$ … multiplicities)
- **Examp** …

… 0.

Research task Copy the sequence and put it into OEIS

$\otimes$ cat counts – char zero



- $b_n = b_n^{\Gamma,V}$=number of indecomposable summands of $V^{\otimes n}$ (with multiplicities)
- **Example** $\Gamma = SL_2$, $\mathbb{K} = \mathbb{C}$, $V = Sym^2\mathbb{C}^2$ (the 3d simple), then

$$\{1, 1, 3, 7, 19, 51, 141, 393, 1107, 3139, 8953\}, \quad b_n \text{ for } n = 0, ..., 10.$$

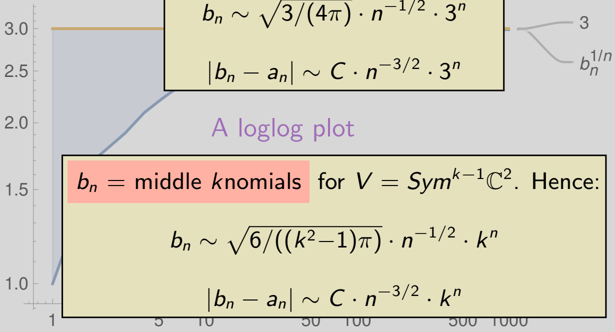
Research task Copy the sequence and put it into OEIS

⊗ cat counts – char zer

b_n = middle trinomials Hence:

$$b_n \sim \sqrt{3/(4\pi)} \cdot n^{-1/2} \cdot 3^n$$

$$|b_n - a_n| \sim C \cdot n^{-3/2} \cdot 3^n$$



A loglog plot

b_n = middle $k$nomials for $V = Sym^{k-1}\mathbb{C}^2$. Hence:

$$b_n \sim \sqrt{6/((k^2-1)\pi)} \cdot n^{-1/2} \cdot k^n$$

$$|b_n - a_n| \sim C \cdot n^{-3/2} \cdot k^n$$

- $b_n = b_n^{\Gamma,V}$=num … (with multiplicities)
- **Example** $\Gamma = S$ … , then

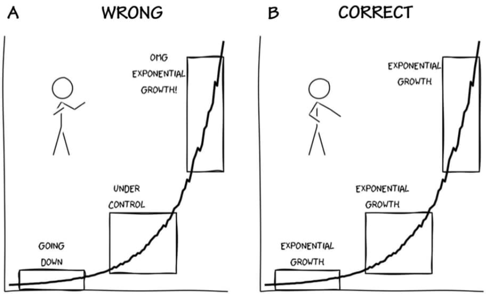
$\{1, 1, 3, 7,$ … or $n = 0, ..., 10$.

Conjecture (for b_n)

Dominating growth is for $\beta = \dim_{\mathbb{K}} V$
Subexponential factor n^{τ} only depends on Γ
h is a scalar

Research task Copy the sequence and put it into OEIS

⊗ cat counts – char zero



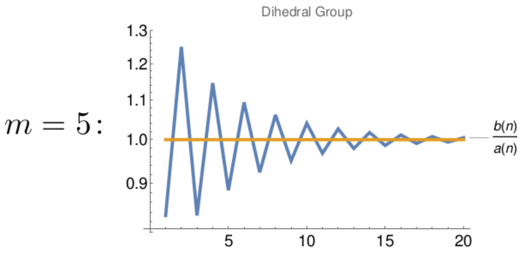
- **Theorem A** The dominating growth is always the dimension (proven for all semigroup superschemes Γ, all fields, all fd reps V)
- **Theorem B** n^{τ} only depends on Γ (proven for all groups, characteristic zero fields, all fd reps V)
- **Theorem C** h takes only finitely many values (proven for all groups, characteristic zero fields, all fd reps V)

Example

Dihedral group of order 10, $\mathbb{K}=\mathbb{C}$, V =any simple 2d

$$b_n \sim \left(\tfrac{7}{10}+\tfrac{1}{5}(-1)^n\right)\cdot n^0\cdot 2^n$$

$m = 5$:



▶ Theor... ...ven for all semigr...

▶ Theor... ...eristic zero fields, ...

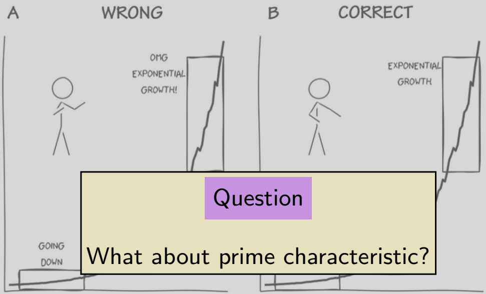
Comments (char zero)

For Γ simple reductive group, $\tau = -\#\text{pos. roots}/2$
and h =scalar given by closed formula, variance=closed formula

For Γ finite group, $\tau = 0$
and h+variance computable from the character table

▶ The above is due to many people, e.g.:
Biane, Bryant–Kovács, Coulembier–Etingof–Ostrik, Lacabanne–Vaz, He,...

⊗ cat counts – char zero



Question

What about prime characteristic?

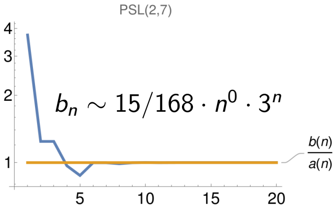
- Theorem A The dominating growth is always the dimension (proven for all semigroup superschemes Γ, all fields, all fd reps V)
- Theorem B n^{τ} only depends on Γ (proven for all groups, characteristic zero fields, all fd reps V)
- Theorem C h takes only finitely many values (proven for all groups, characteristic zero fields, all fd reps V)

$PSL_2(\mathbb{F}_7)$

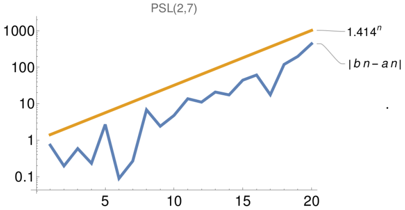
$\mathbb{K} = \mathbb{F}_2$:

V=any 3d simple

$b_n \sim 15/168 \cdot n^0 \cdot 3^n$



Variance:



- Γ = a finite group, $\mathbb{K}$ = any ground field, V = any fin dim Γ-rep
- **Coulembier–Etingof–Ostrik, Lacabanne–Vaz, He ∼2024** This works as in char zero

⊗ cat counts – prime char

- Done char zero: all groups; char p: finite groups
- Next $\Gamma = SL_2(\bar{\mathbb{F}}_p)$, $\mathbb{K} = \bar{\mathbb{F}}_p$
- We will see a remarkable complexity jump

One finds fractals in asymptotic counting problems in monoidal categories defined over fields of prime characteristic

Next Two non-monoidal primers due to:

Haboush ∼1980 (first)
Carter–Cline ∼1976 (second)
Coulembier–Etingof–Ostrik ∼2024 (put together)

After that The monoidal case due to:

Larsen, Coulembier–Etingof–Ostrik ∼2024

- Done char zero: all groups; char p: finite groups
- Next $\Gamma = SL_2(\bar{\mathbb{F}}_p)$, $\mathbb{K} = \bar{\mathbb{F}}_p$
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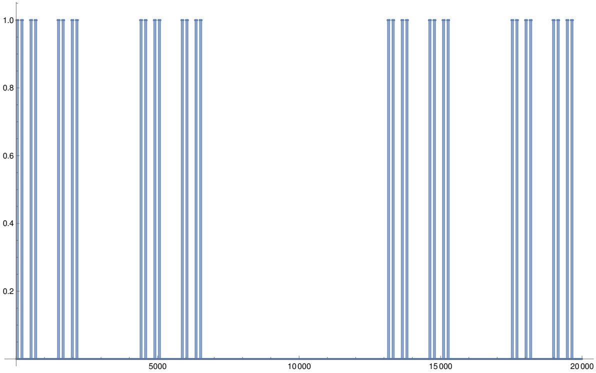
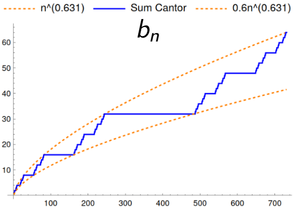
$\otimes$ cat counts – prime char

ca_n :



- For $p = 3$, let $L_{-1/2}$ be the simple rep of highest weight $-1/2$
- ca_n = the dimension of its weight space of weight $-1/2 - n$
- $b_n = \sum_{k=0}^{n} ca_k$, which quantifies the growth of $L_{-1/2}$ satisfies $h(n) \cdot n^{\tau} \cdot \beta^n$ with $\beta = 1$ Recall: if you see the above, take the sum

⊗ cat counts – prime char



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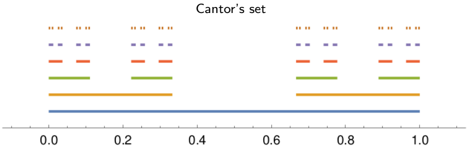

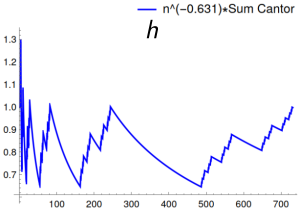

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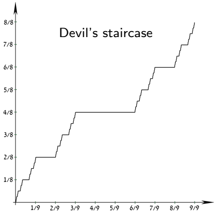
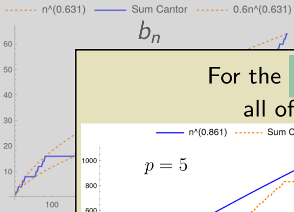


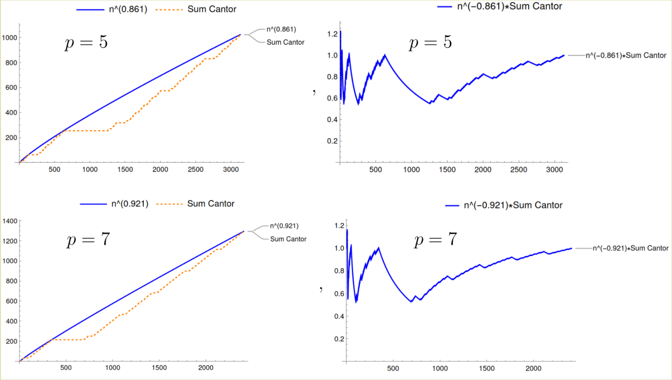
- New 1 $\tau = \log_3 2 =$ dim of Cantor set ≈ 0.631
- New 2 h is insane: it approaches a periodic function akin to devil's staircase

⊗ cat counts – prime char



Cantor's set

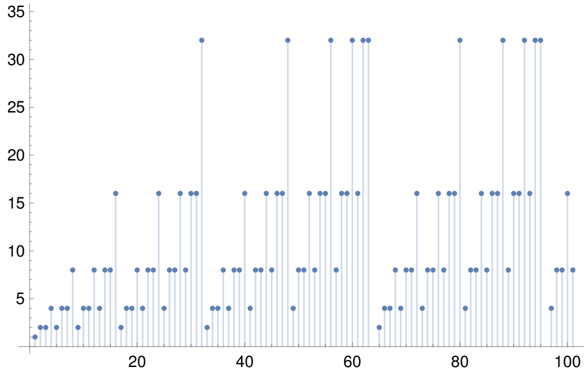
For the transcendental $\tau = \log_p(p-1)$
all of this works for general $p > 2$



- **New 1** $\tau = \log_3 2 =$ dim of Cantor set ≈ 0.631
- **New 2** h is insane: it approaches a periodic function akin to devil's staircase

⊗ cat counts – prime char

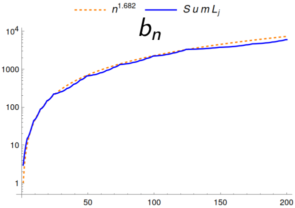
dim L_n :



- For $p = 2$, let L_n be the simple rep of highest weight $n \in \mathbb{N}$
- $\dim L_n$ = the dimension of it
- $b_n = \sum_{k=0}^{n} \dim L_k$, which quantifies the growth of L_n satisfies $h(n) \cdot n^{\tau} \cdot \beta^{n}$ with $\beta = 1$ Recall: if you see the above, take the sum

⊗ cat counts – prime char

$p = 5$:



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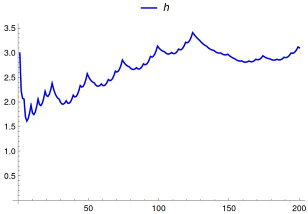
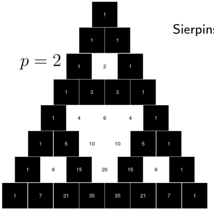


$p = 2$



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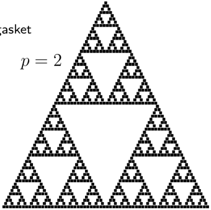
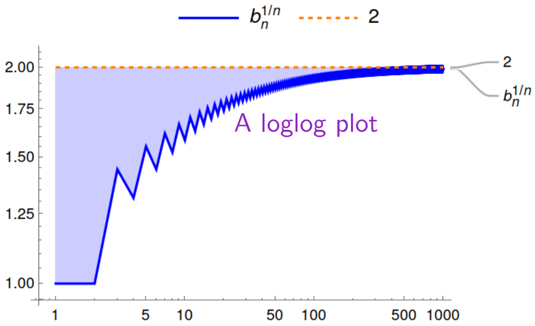
Sierpinski's gasket

$p = 2$



- New 1 $\tau = 1 + \log_p \frac{p+1}{2}$ = dim of Sierpinski's gasket ≈ 1.682 for $p = 5$
- New 2 h is again insane

$\otimes$ cat counts – prime char



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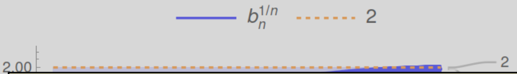
- **Example** $\Gamma = SL_2$, $\mathbb{K} = \bar{\mathbb{F}}_p$, $V = \bar{\mathbb{F}}_p^2$ (vector rep), then

$$\{1, 1, 1, 3, 3, 9, 9, 29, 29, 99, 99\}, \quad b_n \text{ for } n = 0, ..., 10.$$

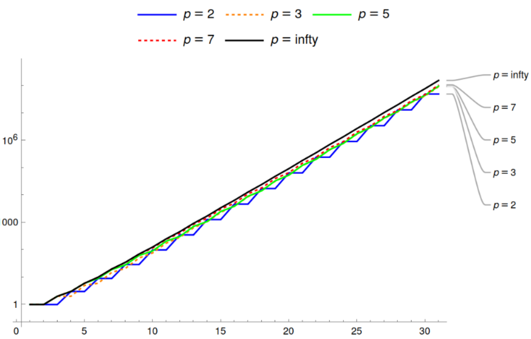
Research task Copy the sequence and put it into OEIS

⊗ cat counts – prime char



b_n = new sequence

$$b_n \sim h(n) \cdot n^{1/2 \log_p \frac{p+1}{2} - 1} \cdot 2^n$$



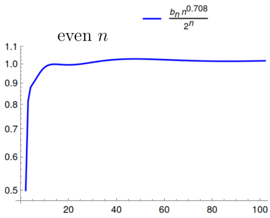
- $b_n = b_n^{\Gamma, V} =$ … ith multiplicities)
- **Example Γ** …

…, 10.

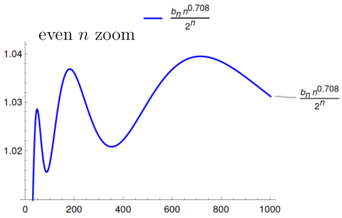
Research task Copy the sequence and put it into OEIS

$\otimes$ cat counts – prime char

$h, p = 2$:



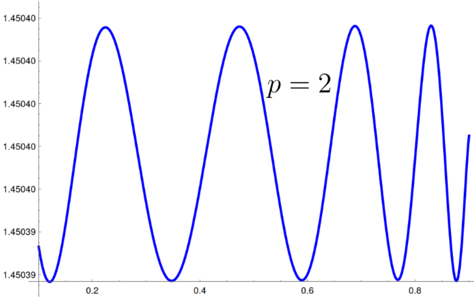
$h, p = 2$:



- **New 1** $\tau = 1/2 \log_p \frac{p+1}{2} - 1 = \text{dim of ???} \approx -0.708$ for $p = 2$
- **New 2** h is again insane

⊗ cat counts – prime char

zoom in $h, p = 2$:



- **h is really insane** It has ∞ many nonzero Fourier coefficients L_n (highly oscillating)
- Some **analytic number theory** going on:
  - The L_n involve the (Hurwitz) zeta and Gamma function
  - There are functional equations akin to Mahler functions and Dirichlet's L-function

⊗ cat counts –

$$b_n \sim h(n) \cdot n^{\tau} \cdot \beta^{n}$$

Recall the char zero results:

- Theorem A The dominating growth is always the dimension (proven for all semigroup superschemes Γ, all fields, all fd reps V)
- Theorem B n^{τ} only depends on Γ (proven for all groups, characteristic zero fields, all fd reps V)
- Theorem C h takes only finitely many values (proven for all groups, characteristic zero fields, all fd reps V)

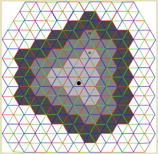
Wannabe theorem 1 in prime characteristic

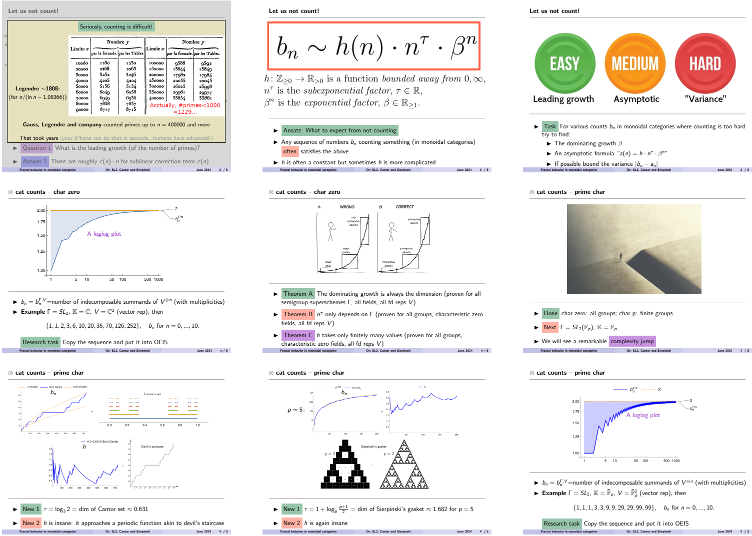
Theorem B also holds (Theorem A is always true, Theorem C is false)

- h is really ins… …nts L_n (highly oscillating)
- Some analyti…
  - The L_n i… …on
  - There are… …ns and Dirichlet's L-functio…

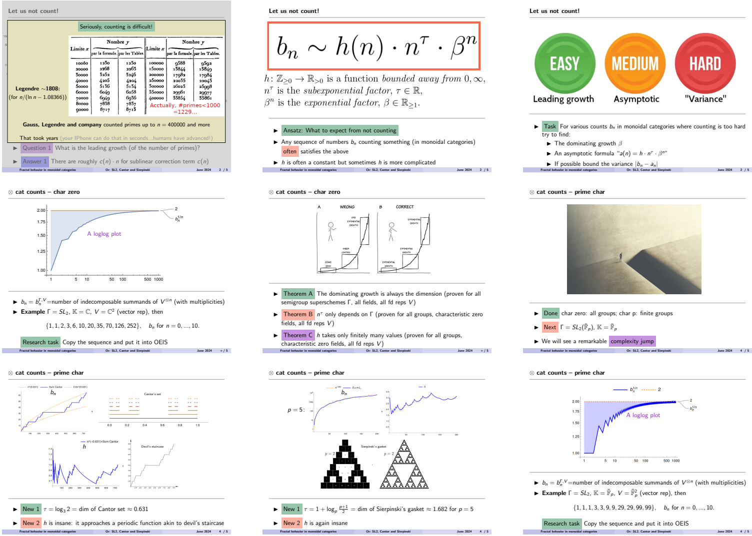
Wannabe theorem 2

Something similar works for the Hecke category





There is still much to do...



Thanks for your attention!