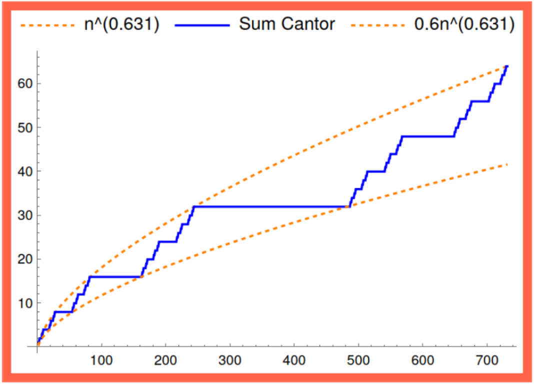
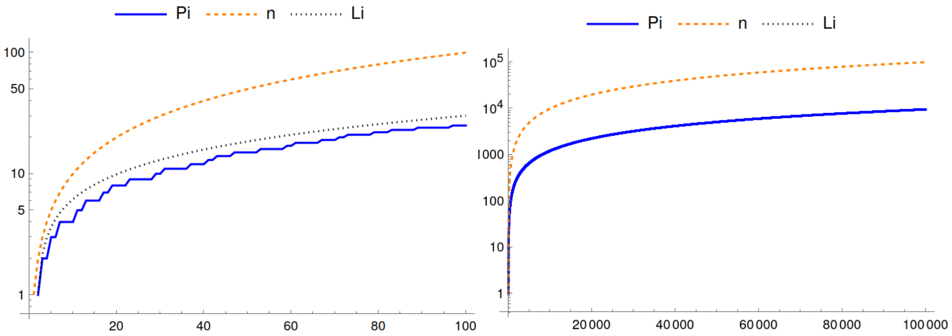
Analytic theory of monoidal categories

Or: Strategies to avoid counting



This is part 1

Let us not count!



- Prime number function $\pi(n)$ $= \#$ primes $\leq$ n
- Counting primes is very tricky as primes “pop up randomly”
- Question 1 What is the leading growth (of the number of primes)?
- Answer 1 There are roughly $c(n) \cdot n$ for sublinear correction term $c(n)$

Let us not count!

Seriously, counting is difficult!

Legendre ∼1808:
(for $n/(\ln n - 1.08366)$)

Limite x	Nombre y par la formule.	Nombre y par les Tables.	Limite x	Nombre y par la formule.	Nombre y par les Tables.
10000	1230	1230	100000	9588	9592
20000	2268	2263	150000	13844	13849
30000	3252	3246	200000	17982	17984
40000	4205	4204	250000	22035	22045
50000	5136	5134	300000	26023	25998
60000	6049	6058	350000	29961	29977
70000	6949	6936	400000	33854	33861
80000	7838	7837			
90000	8717	8713			

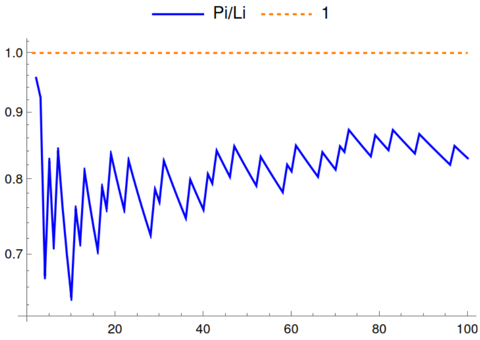
Acctually, #primes<1000 =1229...

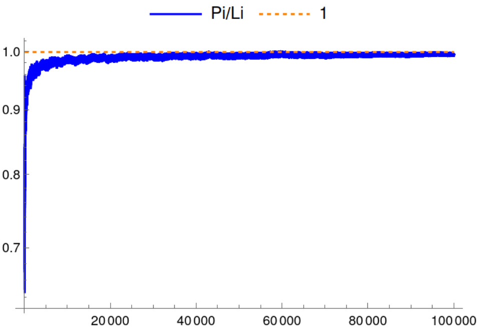
Gauss, Legendre and company counted primes up to $n = 400000$ and more

That took years (your IPhone can do that in seconds...humans have advanced!)

- Question 1 What is the leading growth (of the number of primes)?
- Answer 1 There are roughly $c(n) \cdot n$ for sublinear correction term $c(n)$

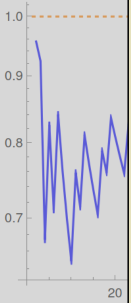
Let us not count!





- Asymptotically equal $f \sim g$ if $\lim_{n\to\infty} f(n)/g(n) \to 1$
- Logarithmic integral $Li(x) = \int_2^x 1/\ln(t)dt$
- Question 2 What is the growth (of the number of primes) asymptotically?
- Answer 2 We have $\pi(n) \sim n/\log(n) \sim Li(n)$

Let us n



How good is this not counting?

Riemann ∼1859 calculates "the variance":

VII.

Ueber die Anzahl der Primzahlen unter einer gegebenen Grösse.

(Monatsberichte der Berliner Akademie, November 1859.)

Durch Einsetzung dieser Werthe in den Ausdruck von $f(x)$ erhält man

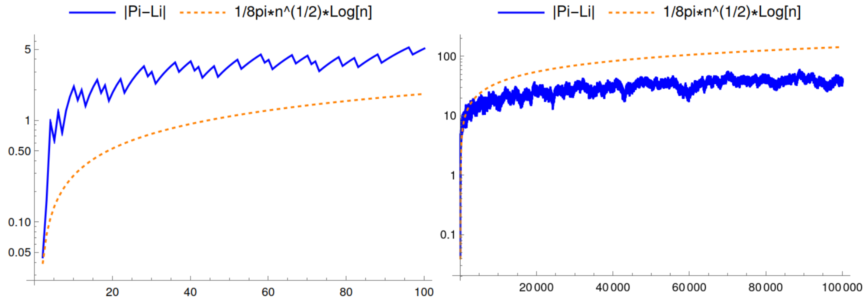
$$f(x) = Li\,(x) - \Sigma^{\alpha}\left(Li\,(x^{\frac{1}{2}+\alpha i}) + Li\,(x^{\frac{1}{2}-\alpha i})\right) + \int_x^{\infty} \frac{1}{x^2-1}\,\frac{dx}{x \log x} + \log \xi\,(0),$$

wenn in Σ^{α} für α sämmtliche positiven (oder einen positiven reellen Theil enthaltenden) Wurzeln der Gleichung $\xi\,(\alpha) = 0$, ihrer Grösse nach geordnet, gesetzt werden. Es lässt sich, mit Hülfe einer genaueren Discussion der Function ξ, leicht zeigen, dass bei dieser Anordnung der Werth der Reihe

f is essentially the prime counting function π

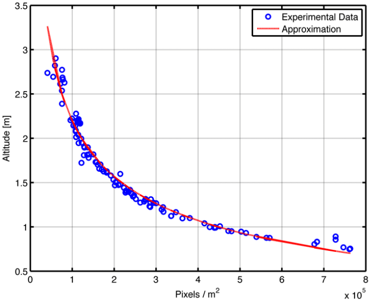
- Asy
- Log
- Que ... tically?
- Ans

Let us not count!



- Asymptotically equal **does not** imply that the difference is good
- $|f(n) - g(n)|$ is a measurement of how good the approximation is
- **Question 3** What is variance from the expected value $(Li(n))$?
- **Conjectural answer 3** We have $|\pi(n) - Li(n)| \in O(n^{1/2}\log n)$ or $|\pi(n) - Li(n)| \leq \frac{1}{8\pi} n^{1/2} \log n$ (for $n \geq 2657$)

Let us not count!



- Observation Precise results are often out of reach but approximate answers are easy to get and beautiful
- These ideas are ubiquitous in discrete math, e.g. in number theory, combinatorics, graph theory, ...
- Idea Do the same in (rep theory + category theory) = monoidal categories

Let us not count!

Today

(1) Motivating examples

After today

We focus on one specific problem:

Counting summands in tensor powers

Key examples are:

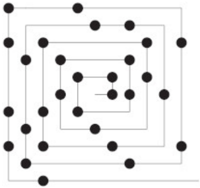
(2) Finite groups

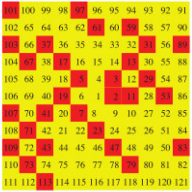
(3) SL_2 + (4) dual problem

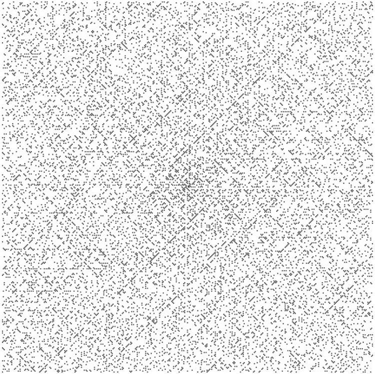
(5) General, including the Hecke category

- Observation Pr … approximate answers are easy to get a …
- These ideas are … mber theory, combinatorics, g …
- Idea Do the sa … monoidal categories

Not counting in graph theory

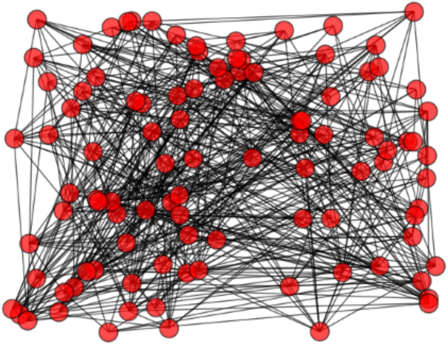






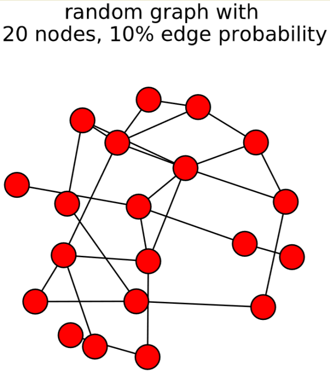
- Prime numbers appear essentially randomly
- Zooming out, they mostly look like noise
- However, also many patterns can be observed

Not counting in graph theory



- Random graphs = choose edges randomly = “average graphs”
- Zooming out, they mostly look like noise
- However, also many patterns can be observed

Not counting in



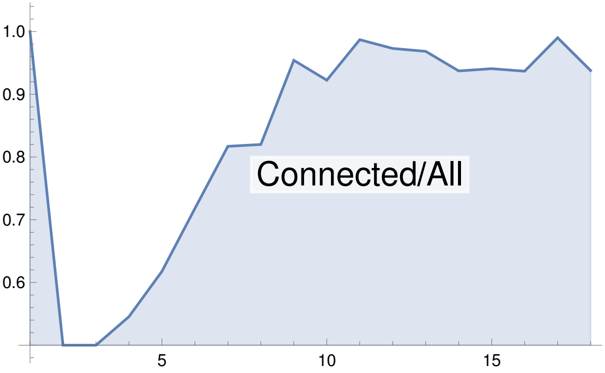
One studies random graphs $G_{n,p}$ for $n = |V| \gg 0$
with p=probability to connect vertices

Asymptotically many patterns arise

Example Almost all random graphs are connected

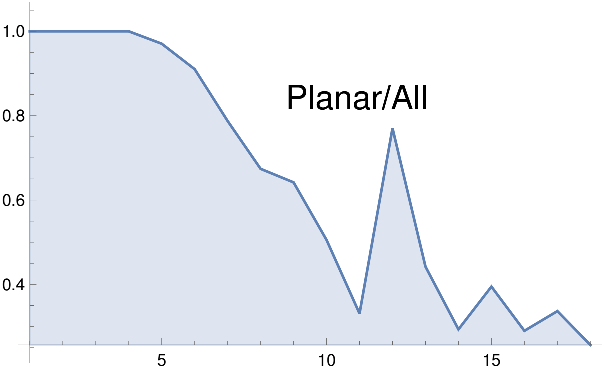
- Random gra ... hs"
- Zooming ou
- However, also many patterns can be observed

Not counting in graph theory



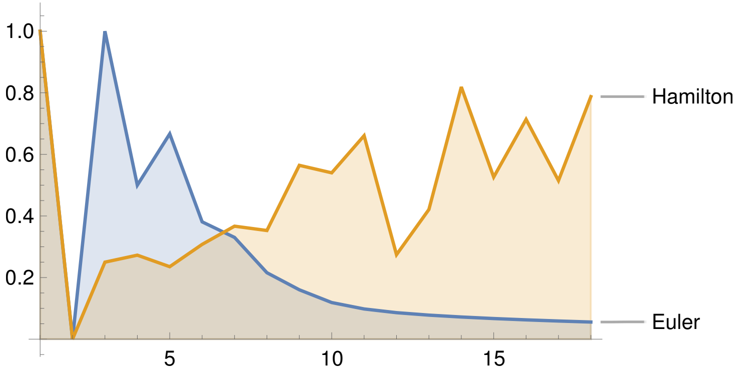
- Fact Most graphs have many edges
- Random graphs are almost always connected (=almost all graphs are connected)
- Above # connected graphs / # all graphs

Not counting in graph theory



- Fact Most graphs have many edges
- Almost no graph is planar
- Above # planar graphs / # all graphs

Not counting in graph theory



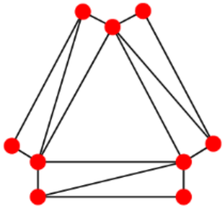
- Fact Most graphs have many edges
- Almost all/no graph is Hamiltonian/Eulerian
- Above # connected Hamil resp. Euler / # all graphs

Not counting in graph theory

$\lvert \mathrm{Aut}\,(G) \rvert$	OEIS	counts of graphs with 1, 2, ... nodes
1	A003400	0, 0, 0, 0, 0, 8, 152, 3696, 135004, ...
2	A075095	0, 2, 2, 3, 11, 46, 354, 4431, 89004, ...
3		0, 0, 0, 0, 0, 0, 0, 0, 4, ...
4	A075096	0, 0, 0, 2, 6, 36, 248, 2264, 31754, ...
6	A075097	0, 0, 2, 2, 2, 8, 38, 252, 3262, ...
8	A075098	0, 0, 0, 2, 4, 14, 74, 623, 7003, ...

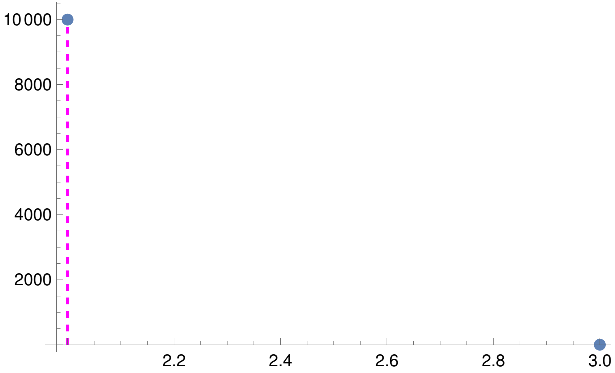
smallest cyclic group graph

$(n = 9, \lvert Aut \rvert = 3)$



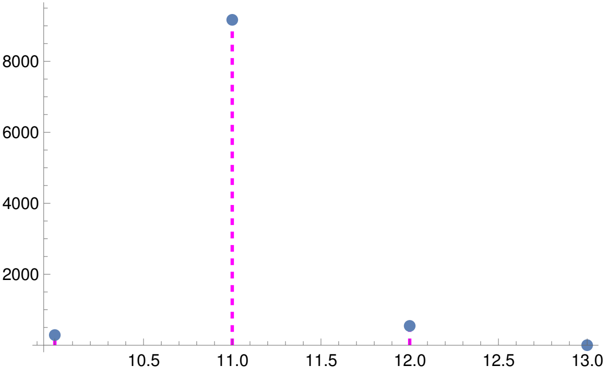
- $Aut(G)$ = group of automorphisms of a graph
- Graph automorphisms keep adjacency so random appearing edges are tricky
- Theorem Almost all graphs have trivial automorphism group

Not counting in graph theory



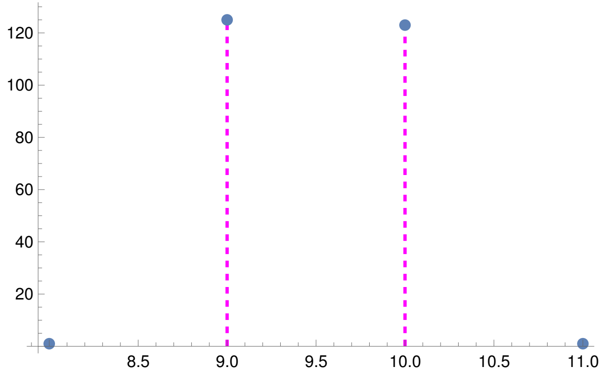
- Above The diameters of 10000 random coin flip graphs with 50 vertices
- Note the clustering
- Easy Almost all $G_{n,p}$ have $d(G_{n,p}) = 2$ (here and later $0 < p \leq 1$)

Not counting in graph theory



- Above Clique number cl of 10000 $G_{50,1/2}$
- There seems to be a peak at one value
- Indeed, the clique number satisfies $cl(G_{n,p}) \approx 2\log_{1/p}(n)$

Not counting in graph theory



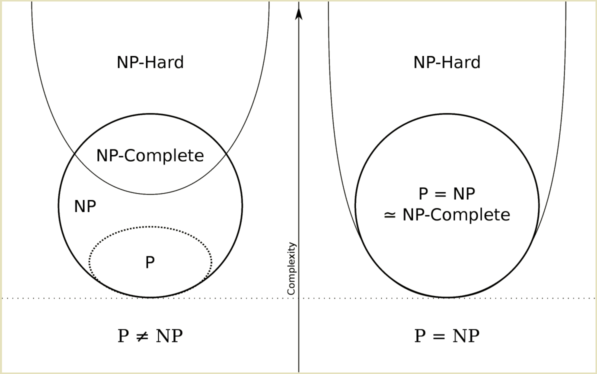
- Above χ of 250 $G_{50,1/2}$
- There seems to be a concentration around one or two values $\approx n/2\log_2(n)$
- Easy The coloring number χ satisfies $\chi(G_{n,p}) \approx n/2\log_{1/(1-p)}(n)$

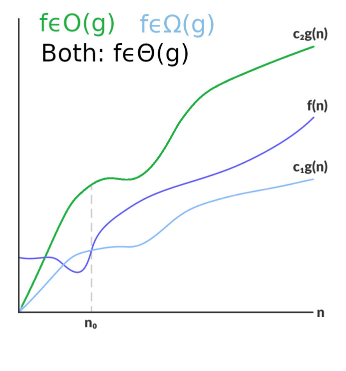
Summary

"On average" and "asymptotic" answers might be nice
even if the precise results are very difficult

Example

Finding the chromatic number χ is NP-hard
The fastest known algorithms to find colorings are $\approx$ in $O(n \cdot 2^n)$
But $\chi(G_{n,p}) \approx n/2 \log_{1/(1-p)} n$ is easy to get





Not counting in algebra

$gnu(n)$:

1 1	1 2	1 3	2 4	1 5	2 6	1 7	5 8	2 9	2 10	1 11	5 12
1 13	2 14	1 15	14 16	1 17	5 18	1 19	5 20	2 21	2 22	1 23	15 24
2 25	2 26	5 27	4 28	1 29	4 30	1 31	51 32	1 33	2 34	1 35	14 36
1 37	2 38	2 39	14 40	1 41	6 42	1 43	4 44	2 45	2 46	1 47	52 48
2 49	5 50	1 51	5 52	1 53	15 54	2 55	13 56	2 57	2 58	1 59	13 60
1 61	2 62	4 63	267 64	1 65	4 66	1 67	5 68	1 69	4 70	1 71	50 72
1 73	2 74	3 75	4 76	1 77	6 78	1 79	52 80	15 81	2 82	1 83	15 84
1 85	2 86	1 87	12 88	1 89	10 90	1 91	4 92	2 93	2 94	1 95	231 96
1 97	5 98	2 99	16 100	1 101	4 102	1 103	14 104	2 105	2 106	1 107	45 108
1 109	6 110	2 111	43 112	1 113	6 114	1 115	5 116	4 117	2 118	1 119	47 120
2 121	2 122	1 123	4 124	5 125	16 126	1 127	2328 128	2 129	4 130	1 131	10 132
1 133	2 134	5 135	15 136	1 137	4 138	1 139	11 140	1 141	2 142	1 143	197 144

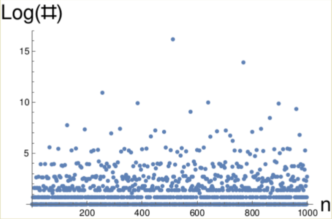
- $gnu(n)$ = number of groups of order n
- Example $gnu(prime) = 1$, but getting other values is very hard
- Surprise One can prove nontrivial facts about $gnu(n)$

Not counting in algebra

Finite groups are kind of random...

A000001 Number of groups of order n.
(Formerly M0098 N0035)

0, 1, 1, 1, 2, 1, 2, 1, 5, 2, 2, 1, 5, 1, 2, 1, 14, 1, 5, 1, 5, 2, 2, 1, 15, 2, 2, 5, 4, 1, 4, 1, 51, 1, 2, 1, 14, 1, 2, 2, 14, 1, 6, 1, 4, 2, 2, 1, 52, 2, 5, 1, 5, 1, 15, 2, 13, 2, 2, 1, 13, 1, 2, 4, 267, 1, 4, 1, 5, 1, 4, 1, 50, 1, 2, 3, 4, 1, 6, 1, 52, 15, 2, 1, 15, 1, 2, 1, 12, 1, 10, 1,



- Pattern in a sea of randomness The 11758615 groups of order <1000 are swamped by the 10494213 of order 512
- However determining $gnu(n)$ precisely (even for prime powers) is very difficult
- So counting fails

Not countin

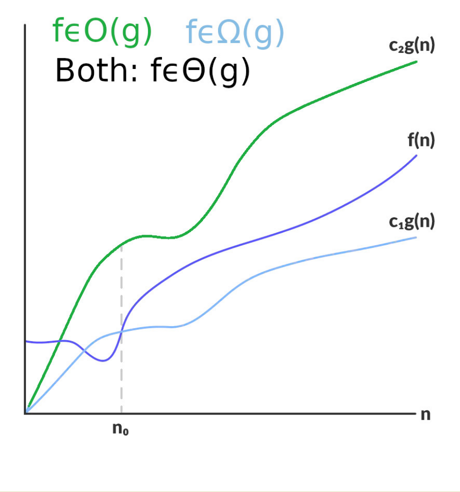
Theorem

Sims ∼1964 For a prime we have $gnu(p^n) \sim p^{2n^3/27+O(n^{8/3})}$

A000001	Num (Form

0, 1, 1, 1, 2, 1 … , 5, 4, 1, 4, 1,
51, 1, 2, 1, 14, … 2, 2, 1, 13, 1,
2, 4, 267, 1, 4, … 1, 12, 1, 10, 1,

Reminder on capital O notation



- Pattern in a se … order <1000 are swamped by th
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Not countin

Theorem

Sims ∼1964 For a prime we have $gnu(p^n) \sim p^{2n^3/27+O(n^{8/3})}$

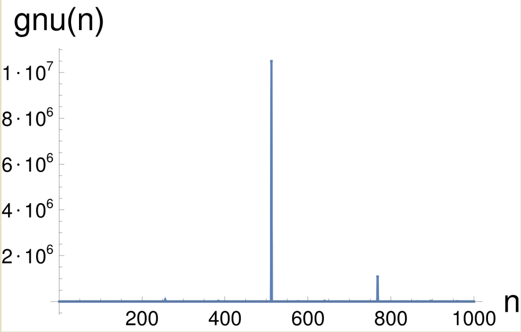
Theorem

Kruse–Price ∼1970, Blackburn–McLean ∼2021
For a prime we have 'isoclasses of rings of size'$(p^n) \sim p^{4n^3/27+O(n^{8/3})}$

A0

Folklore conjecture

Almost all groups are of order 2^n



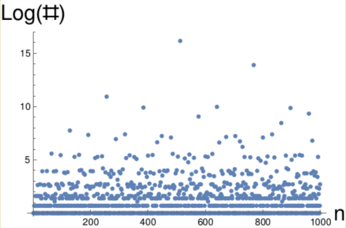
- Pattern in a … rder <1000 are swamped by
- However de … rs) is very difficult
- So counting

Not counting in algebra

Finite groups are kind of random...

A000001 Number of groups of order n.
(Formerly M0098 N0035)

0, 1, 1, 1, 2, 1, 2, 1, 5, 2, 2, 1, 5, 1, 2, 1, 14, 1, 5, 1, 5, 2, 2, 1, 15, 2, 2, 5, 4, 1, 4, 1,
51, 1, 2, 1, 14, 1, 2, 2, 14, 1, 6, 1, 4, 2, 2, 1, 52, 2, 5, 1, 5, 1, 15, 2, 13, 2, 2, 1, 13, 1,
2, 4, 267, 1, 4, 1, 5, 1, 4, 1, 50, 1, 2, 3, 4, 1, 6, 1, 52, 15, 2, 1, 15, 1, 2, 1, 12, 1, 10, 1,

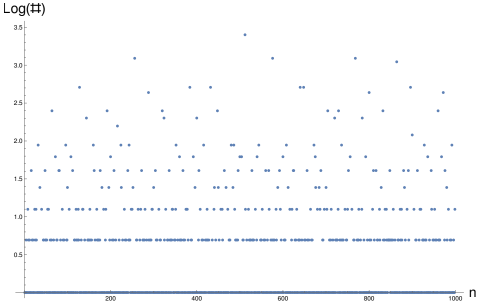


- Summary
- The gnu function $gnu(n)$ = number of different groups of size n
- Problem We know next to nothing about $gnu(n)$, but the ‘growth’ is fast

Not counting in algebra

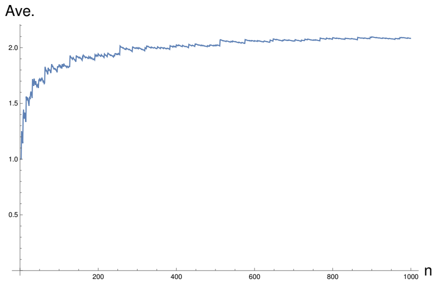
A000688 Number of Abelian groups of order n; number of factorizations of n into prime powers. 129
(Formerly M0064 N0020)

1, 1, 1, 2, 1, 1, 1, 3, 2, 1, 1, 2, 1, 1, 1, 5, 1, 2, 1, 2, 1, 1, 1, 3, 2, 1, 3, 2, 1, 1, 1, 7, 1,
1, 1, 4, 1, 1, 1, 3, 1, 1, 1, 2, 2, 1, 1, 5, 2, 2, 1, 2, 1, 3, 1, 3, 1, 1, 1, 2, 1, 1, 2, 11, 1,
1, 1, 2, 1, 1, 1, 6, 1, 1, 2, 2, 1, 1, 1, 5, 5, 1, 1, 2, 1, 1, 1, 3, 1, 2, 1, 2, 1, 1, 1, 7, 1, 2,
2, 4, 1, 1, 1, 3, 1, 1, 1 (list; graph; refs; listen; history; text; internal format)



- Abelian groups of order n = symmetries with n commuting operations
- The agnu function $agnu(n)$ = number of different abelian groups of size n
- Task Describe $agnu(n)$

Not counting in algebra



- The average number of abelian groups of a given order is

$$\prod_{j\geq 2}\zeta(j)\approx 2.29485659$$

- ζ is the (Riemann) zeta function
- Average is in the sense of arithmetic mean: $\frac{1}{n}\sum_{k=1}^{n} agnu(k)$

Not counting in algebra

Theorem

Knopfmacher ∼1972 The average number of semisimple rings is

$$\prod_{rm^2>1} \zeta(rm^2) \approx 2.49961611$$

Semisimple = matrix rings

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ \vdots \\ m \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & \cdots & n \end{array} \\ \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ a_{31} & a_{32} & \dots & a_{3n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \end{array}$$

- The a
- ζ is the (Riemann) zeta function
- Average is in the sense of arithmetic mean: $\frac{1}{n}\sum_{k=1}^{n} agnu(k)$

Not counting in representation theory

S_3:

```
-----------------
Class |   1  2  3
Size  |   1  3  2
Order |   1  2  3
-----------------
p  =  2   1  1  3
p  =  3   1  2  1
-----------------
X.1   +   1  1  1
X.2   +   1 -1  1
X.3   +   2  0 -1
```

, S_4:

```
-----------------------
Class |   1  2  3  4  5
Size  |   1  3  6  8  6
Order |   1  2  2  3  4
-----------------------
p  =  2   1  1  1  4  2
p  =  3   1  2  3  1  5
-----------------------
X.1   +   1  1  1  1  1
X.2   +   1  1 -1  1 -1
X.3   +   2  2  0 -1  0
X.4   +   3 -1 -1  0  1
X.5   +   3 -1  1  0 -1
```

- Character table = prototypical rep theory
- Example The character tables of S_3 and S_4 created with Magma
- What do we see? Rows = simple characters, columns = conjugacy classes, size = number of elements, order = order of elements; rest (Schur indicator, power map) = not important today

Not counting in representation theory

D_4:

```
------------------------
Class |   1  2  3  4  5
Size  |   1  1  2  2  2
Order |   1  2  2  2  4
------------------------
p  =  2   1  1  1  1  2
------------------------
X.1    +   1  1  1  1  1
X.2    +   1  1 -1  1 -1
X.3    +   1  1  1 -1 -1
X.4    +   1  1 -1 -1  1
X.5    +   2 -2  0  0  0
```

, Q_8:

```
------------------------
Class |   1  2  3  4  5
Size  |   1  1  2  2  2
Order |   1  2  4  4  4
------------------------
p  =  2   1  1  2  2  2
------------------------
X.1    +   1  1  1  1  1
X.2    +   1  1 -1  1 -1
X.3    +   1  1  1 -1 -1
X.4    +   1  1 -1 -1  1
X.5    -   2 -2  0  0  0
```

- Characters do not determine finite groups
- First example The dihedral group D_4 (eight elements) and Q_8 (quaternions)
- Question How good is the character table?

Not counting in representation theory

```
------------------------
Class |   1  2  3  4  5
```

```
------------------------
Class |   1  2  3  4  5
```

Some patterns

Frobenius? ∼1900 Abelian groups are determined by their characters
(Even better: two groups with the same characters have isomorphic abelianizations)

Oyama ∼1964 Symmetric groups are determined by their characters

Many people ∼1960++ Nonabelian simple groups are determined by their characters

```
X.3    +    1  1  1 -1 -1
X.4
X.5
```

```
X.3    +    1  1  1 -1 -1
            1  1
            0  0
```

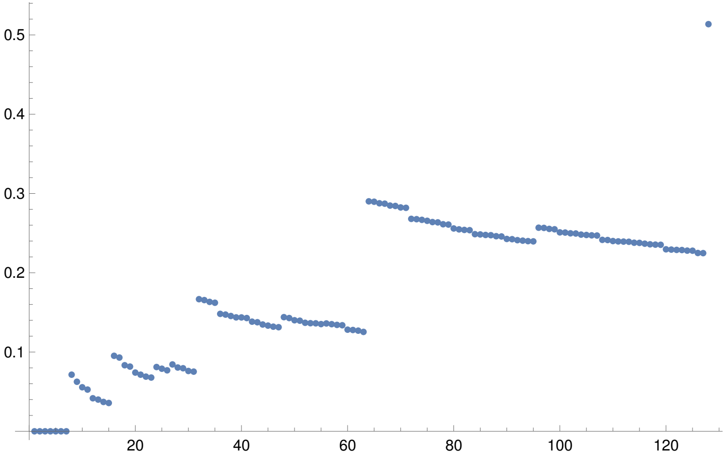
Problem **Brauer ∼1963**

Classify the groups that are determined by their characters

This is probably out of reach

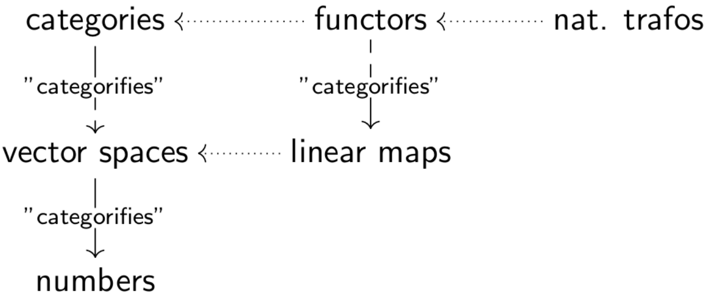
- Characte
- First example The dihedral group D_4 (eight elements) and Q_8 (quaternions)
- Question How good is the character table?

Not counting in representation theory



- Folklore conjecture Almost all group are not determined by characters
- Above % of groups $\leq n$ not determined by their characters

Not counting in representation theory



- 'Categorification' Are groups determined by their representation categories?
- In formulas (complex coefficients)

$$\big(\mathbf{Rep}(G) \cong_{\otimes} \mathbf{Rep}(H)\big) \Rightarrow \big(G \cong H\big)?$$

Such groups are called categorically rigid

Some patterns (all **Etingof–Gelaki ∼2000**)

All groups that are determined by their characters are categorically rigid

All groups of order $\equiv 1$ or 2 or 3 mod 4 are categorically rigid

All groups of order < 64 are categorically rigid

Categorically rigid is independent of the field (as long as $\mathbb{K} = \bar{\mathbb{K}}$)

numbers

- ‘Catego … categories?
- In form …

Theorem **Etingof–Gelaki, Davydov, Izumi–Kosaki ∼2000**

One can classify the groups that are determined by their representation categories

Categorification helps!

Such groups are called categorically rigid

Not counting in representation theory

	n	$n \log_2 n$	n^2	n^3	1.5^n	2^n	$n!$
$n = 10$	< 1 sec	< 1 sec	< 1 sec	< 1 sec	< 1 sec	< 1 sec	4 sec
$n = 30$	< 1 sec	< 1 sec	< 1 sec	< 1 sec	< 1 sec	18 min	10^{25} years
$n = 50$	< 1 sec	< 1 sec	< 1 sec	< 1 sec	11 min	36 years	very long
$n = 100$	< 1 sec	< 1 sec	< 1 sec	1 sec	12,892 years	10^{17} years	very long
$n = 1{,}000$	< 1 sec	< 1 sec	1 sec	18 min	very long	very long	very long
$n = 10{,}000$	< 1 sec	< 1 sec	2 min	12 days	very long	very long	very long
$n = 100{,}000$	< 1 sec	2 sec	3 hours	32 years	very long	very long	very long
$n = 1{,}000{,}000$	1 sec	20 sec	12 days	31,710 years	very long	very long	very long

Polynomial time	Exponential Time
n - Linear Search logn - Binary Search n*n - Insertion Sort n*logn - Merge Sort n*n*n - Matrix Multiplication	2^n - 0/1 knapsack 2^n - Travelling SP 2^n - Sum of Subsets 2^n - Graph Coloring 2^n - Hamilton Cycle

- **Problem** Checking this is very difficult
- **Compare** Computing character tables is $O(\text{poly in } \#G)$

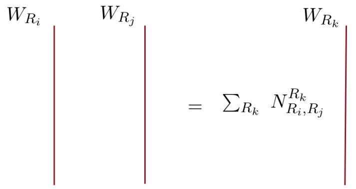
Not counting in re

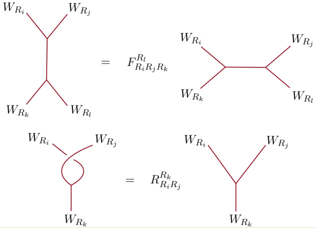
	n!
$n = 10$	4 sec
$n = 30$	10^{25} years
$n = 50$	very long
$n = 100$	very long
$n = 1{,}000$	very long
$n = 10{,}000$	very long
$n = 100{,}000$	very long
$n = 1{,}000{,}000$	very long

n
log
n*
n*
n*

Theorem **Deligne–Milne ∼1982**

If you include symmetry (the 'braiding'), then all groups are determined by **Rep**(G)

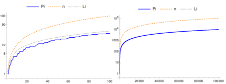




This is however even more difficult to check

- **Problem** Chec
- **Compare** Computing character tables is $O(\text{poly in } \#G)$

Let us not count!



- Prime number function $\pi(n)$ = # primes $\leq$ n
- Counting primes is very tricky as primes "pop up randomly"
- Question 1 What is the leading growth (of the number of primes)?
- Answer 1 There are roughly $c(n)\cdot n$ for sublinear correction term $c(n)$

Let us not count!

Seriously, counting is difficult!

Legendre ~**1808:** (for $n/(\ln n - 1.08366)$)

Limite x	Nombre y par la formule.	Nombre y par les Tables.	Limite x	Nombre y par la formule.	Nombre y par les Tables.
10000	1230	1230	100000	9588	9592
20000	2268	2263	150000	13844	13849
30000	3252	3246	200000	17982	17984
40000	4205	4204	250000	22035	22045
50000	5136	5134	300000	26023	25998
60000	6049	6058	350000	29961	29977
70000	6949	6936	400000	33854	33861
80000	7838	7837			
90000	8717	8713			

Acctually, #primes<1000 =1229...

Gauss, Legendre and company counted primes up to $n = 400000$ and more

That took years (your IPhone can do that in seconds...humans have advanced!)

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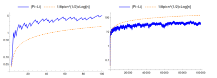

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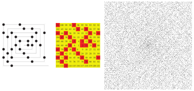
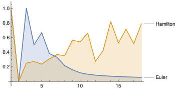
- $|f(n) - g(n)|$ is a measurement of how good the approximation is
- Question 3 What is variance from the expected value ($Li(n)$)?
- Conjectural answer 3 We have $|\pi(n) - Li(n)| \in O(n^{1/2}\log n)$ or $|\pi(n) - Li(n)| \leq \frac{1}{8\pi}n^{1/2}\log n$ (for $n \geq 2657$)

Not counting in graph theory



- Prime numbers appear essentially randomly
- Zooming out, they mostly look like noise
- However, also many patterns can be observed

Not counting in graph theory



- Fact Most graphs have many edges
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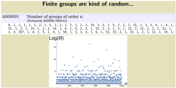
Not counting in algebra



- Pattern in a sea of randomness The 11758615 groups of order <1000 are swamped by the 10494213 of order 512
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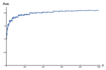
Not counting in algebra



- The average number of abelian groups of a given order is

$$\prod_{j\geq 2}\zeta(j) \approx 2.29485659$$

- ζ is the (Riemann) zeta function
- Average is in the sense of arithmetic mean: $\frac{1}{n}\sum_{k=1}^{n} agnu(k)$

Not counting in representation theory

S_3:

```
-----------------
Class |   1  2  3
Size  |   1  3  2
Order |   1  2  3
-----------------
p  =  2   1  1  3
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-----------------
X.1   +   1  1  1
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X.3   +   2  0 -1
```

, S_4:

```
-----------------------
Class |   1  2  3  4  5
Size  |   1  3  6  8  6
Order |   1  2  2  3  4
-----------------------
p  =  2   1  1  1  4  2
p  =  3   1  2  3  1  5
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```
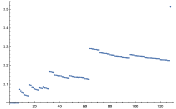

- Character table = prototypical rep theory
- Example The character tables of S_3 and S_4 created with Magma
- What do we see? Rows = simple characters, columns = conjugacy classes, size = number of elements, order = order of elements; rest (Schur indicator, power map) = not important today

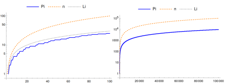
Not counting in representation theory



- Folklore conjecture Almost all group are not determined by characters
- Above % of groups $\leq n$ not determined by their characters

There is still much to do...

Let us not count!



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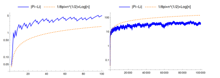

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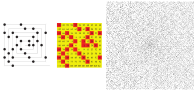

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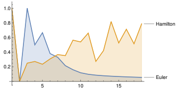

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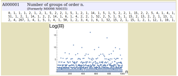
Finite groups are kind of random...



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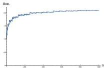

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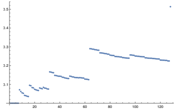

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Thanks for your attention!