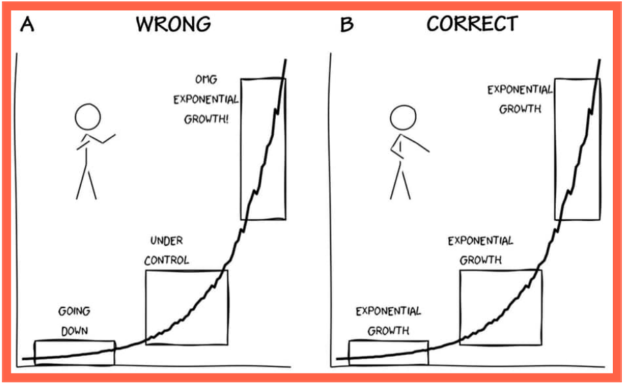
Analytic theory of monoidal categories

Or: Strategies to avoid counting



This is part 2

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Let us not count!

- Γ = something that has a tensor product (more details later)
- $\mathbb{K}$ = any ground field, V = any fin dim Γ-rep
- Problem Decompose $V^{\otimes n}$; note that $\dim_{\mathbb{K}} V^{\otimes n} = (\dim_{\mathbb{K}} V)^n$

Examples of what Γ could be

Any finite group, monoid, semigroup

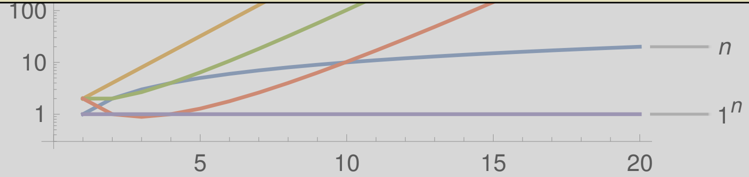
Symmetric groups, alternating groups, cyclic groups, the monster, $GL_N(\mathbb{F}_{p^k})$, ...

Actually any group, monoid, semigroup

$GL_N(\mathbb{C})$, $GL_N(\mathbb{R})$, $GL_N(\bar{\mathbb{F}}_p^k)$, symplectic, orthogonal, braid groups, Thompson groups, ...

Super versions

$GL_{M|N}$, $OSP_{M|2N}$, periplectic, queer, ...



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Examples (that we will touch later)

Up to some slight change of setting we could also include:

Fusion categories or even finite additive Krull–Schmidt monoidal categories

Proj$(G, \mathbb{K})$, **Inj**$(G, \mathbb{K})$, semisimpl. of quantum group reps, Soergel bimodules of finite type, ...

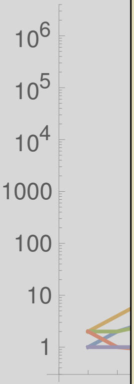
General additive Krull–Schmidt monoidal categories up to one condition (given later)

Rep(GL_n) and friends, quantum group reps, Soergel bimodules of affine type, ...

Most importantly, your favorite example might be included on this list

- Problem Decompose $V^{\otimes n}$; note that $\dim_{\mathbb{K}} V^{\otimes n} = (\dim_{\mathbb{K}} V)^n$

Let us not count!



Let us pause for a second...the setting is way to general!

Decomposing $V^{\otimes n}$ for an arbitrary group is not happening

Better: Let us answer a different question!

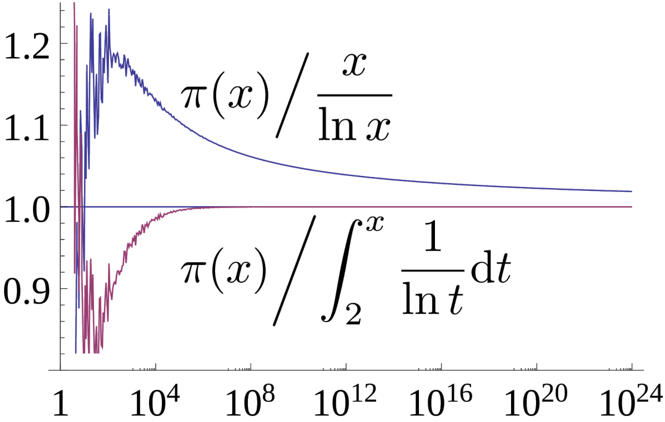
im $V)^n$

lim $V)^n$ / n

lim $V)^n$ / n^2

- Γ = som
- $\mathbb{K}$ = any ground field, V = any fin dim Γ-rep
- **Problem** Decompose $V^{\otimes n}$; note that $\dim_{\mathbb{K}} V^{\otimes n} = (\dim_{\mathbb{K}} V)^n$

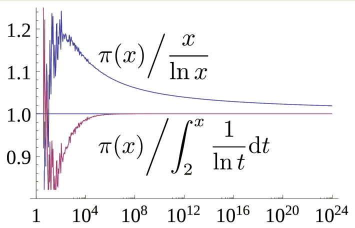
Let us not count!

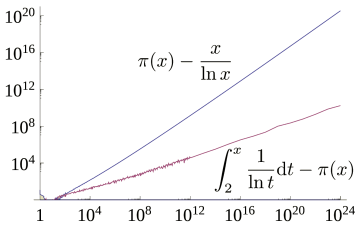


- Counting primes is difficult but...
- **Prime number theorem (many people ∼1793)** #primes $= \pi(n) \sim n/\ln n$

Let us

$\sim$ means asymptotically = ratios are good (not the absolute difference!)





So this is not doing the count!

▶ C

▶ **Prime number theorem (many people ~1793)** #primes $= \pi(n) \sim n/\ln n$

Let

Seriously, counting is difficult!

Legendre ∼1808:
(for $n/(\ln n - 1.08366)$)

Limite x	Nombre y par la formule.	Nombre y par les Tables.	Limite x	Nombre y par la formule.	Nombre y par les Tables.
10000	1230	1230	100000	9588	9592
20000	2268	2263	150000	13844	13849
30000	3252	3246	200000	17982	17984
40000	4205	4204	250000	22035	22045
50000	5136	5134	300000	26023	25998
60000	6049	6058	350000	29961	29977
70000	6949	6936	400000	33854	33861
80000	7838	7837			
90000	8717	8713			

Acctually, #primes<1000 =1229...

Gauss, Legendre and company counted primes up to $n = 400000$ and more

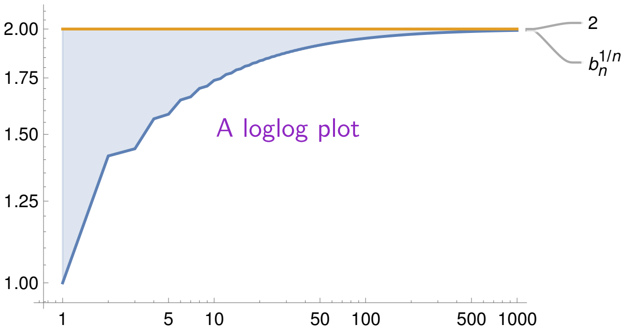
That took years (your IPhone can do that in seconds...humans have advanced!)

"Discrete statements can often be solved approximately"

4

- Counting
- **Prime n** $(n) \sim n/\ln n$

Let us not count!

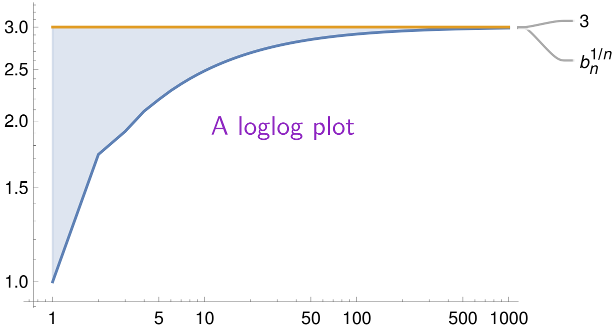


- $b_n = b_n^{\Gamma,V}$=number of indecomposable summands of $V^{\otimes n}$ (with multiplicities)
- **Example** $\Gamma = SL_2$, $\mathbb{K} = \mathbb{C}$, $V = \mathbb{C}^2$, then

$$\{1, 1, 2, 3, 6, 10, 20, 35, 70, 126, 252\}, \quad b_n \text{ for } n = 0, ..., 10.$$

$\lim_{n\to\infty} \sqrt[n]{b_n}$ seems to converge to $2 = \dim_{\mathbb{C}} V$: $\sqrt[1000]{b_{1000}} \approx 1.99265$

Let us not count!



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- **Example** $\Gamma = SL_2$, $\mathbb{K} = \mathbb{C}$, $V = \mathrm{Sym}\,\mathbb{C}^2$ (3d rep), then

$$\{1, 1, 3, 7, 19, 51, 141, 393, 1107, 3139, 8953\}, \quad b_n \text{ for } n = 0, \ldots, 10.$$

$\lim_{n\to\infty} \sqrt[n]{b_n}$ seems to converge to $3 = \dim_{\mathbb{C}} V$: $\sqrt[1000]{b_{1000}} \approx 2.9875$

Let

Observation 1

Whatever is true for SL_2 over $\mathbb{C}$ is true in general, right?

So let us come back to the general setting:
Γ = affine semigroup superscheme
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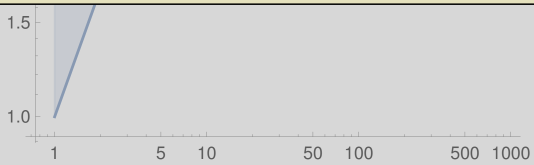

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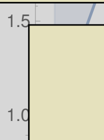

Observation 2

$$b_n b_m \leq b_{n+m} \Rightarrow$$
$$\beta = \lim_{n\to\infty} \sqrt[n]{b_n}$$
is well-defined by a version of Fekete's Subadditive Lemma

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Observation 3

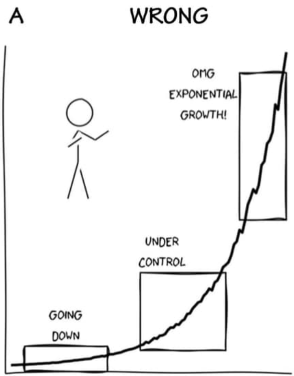
$$1 \leq \beta \leq \dim_{\mathbb{K}} V$$

$\beta = 1 \Leftrightarrow V^{\otimes n}$ for $n \gg 0$ is 'one block'

$\beta = \dim_{\mathbb{K}} V \Leftrightarrow$ summands of $V^{\otimes n}$ for $n \gg 0$ are 'essentially one-dimensional'

$\lim_{n\to\infty} \sqrt[n]{b_n}$ seems to converge to $3 = \dim_{\mathbb{C}} V$: $\sqrt[1000]{b_{1000}} \approx 2.9875$

Let us not count!

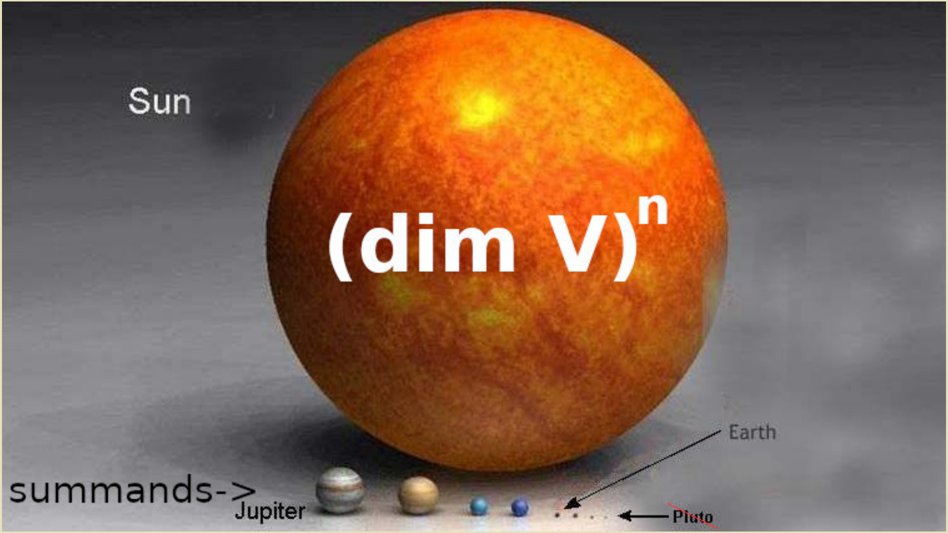


We have

$$\beta = \lim_{n \to \infty} \sqrt[n]{b_n} = \dim_{\mathbb{K}} V$$

Exponential growth is scary

In other words, compared to the size of the exponential growth of $(\dim_{\mathbb{K}} V)^n$
all indecomposable summands are 'essentially one-dimensional'



Let us not count!

Tell you the honest asymptotics

Ansatz! We have

" $b_n \sim h \cdot n^{\tau} \cdot (\dim_{\mathbb{K}} V)^n$ "

for a 'scalar' h and n^{τ}, both killed by $\sqrt[n]{\_}$

Tell you how to prove this

Just kidding; only if you press me...

We have

$$\beta = \lim_{n\to\infty} \sqrt[n]{b_n} = \dim_{\mathbb{K}} V$$

Finite groups - semisimple case

S_3:

```
-----------------
Class |   1  2  3
Size  |   1  3  2
Order |   1  2  3
-----------------
p  =  2   1  1  3
p  =  3   1  2  1
-----------------
X.1    +  1  1  1
X.2    +  1 -1  1
X.3    +  2  0 -1
```

, S_4:

```
-----------------------
Class |   1  2  3  4  5
Size  |   1  3  6  8  6
Order |   1  2  2  3  4
-----------------------
p  =  2   1  1  1  4  2
p  =  3   1  2  3  1  5
-----------------------
X.1    +  1  1  1  1  1
X.2    +  1  1 -1  1 -1
X.3    +  2  2  0 -1  0
X.4    +  3 -1 -1  0  1
X.5    +  3 -1  1  0 -1
```

- Character table = prototypical rep theory
- Example The character tables of S_3 and S_4 created with Magma
- What do we see? Rows = simple characters, columns = conjugacy classes, size = number of elements, order = order of elements; rest (Schur indicator, power map) = not important today

Finite groups - semisimple case

S_3:

```
-----------------
Class |   1  2  3
Size  |   1  3  2
Order |   1  2  3
-----------------
p  =  2   1  1  3
p  =  3   1  2  1
-----------------
X.1   +   1  1  1
X.2   +   1 -1  1
X.3   +   2  0 -1
```

,

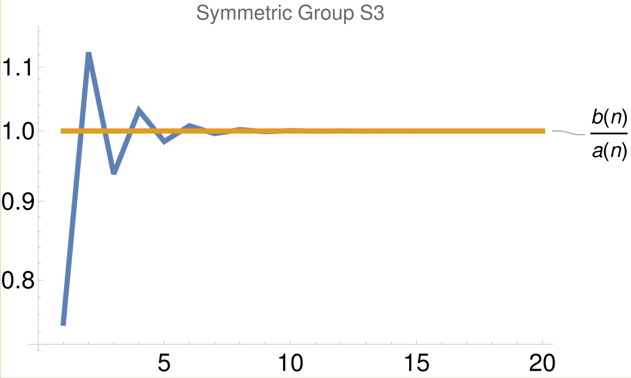
$\cdot$	χ_2	χ_3
χ_2	1	χ_3
χ_3	χ_3	$1+\chi_2+\chi_3$

e.g. $\chi_3^2 = (2,0,-1)^2 = (4,0,1) = (1,1,1) + (1,-1,1) + (2,0,-1) \leftrightsquigarrow [1,1,1]$

- Character ring $[\mathbf{Rep}(G)]$ = elements of the form $\sum_i c_i \chi_i$ for $c_i \in \mathbb{C}$ and multiplication = multiplication of characters
- Crucial fact One can reconstruct $\mathbf{Rep}(G) = \mathbf{Rep}(G, \mathbb{C})$ from its characters
- Example $[\mathbf{Rep}(S_3)]$, $\chi_1 = \text{unit} = 1 \leftrightsquigarrow$ trivial rep rest above

Asymptotic for S_3 and $V = 2$d standard rep

$$\chi_3^n \leftrightsquigarrow [\mathrm{round}(2^n/6), \mathrm{round}(2^n/6), \mathrm{round}(2^n/3)]$$
$$b_n \sim a_n = \tfrac{4}{6} \cdot n^0 \cdot 2^n$$



e.g. $\chi_3^2 =$ ... $\leftrightsquigarrow [1, 1, 1]$

- Charac... $\in \mathbb{C}$ and multipli...
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- Example $[\mathbf{Rep}(S_3)]$, $\chi_1 = \text{unit} = 1 \leftrightsquigarrow$ trivial rep rest above

Finite groups - semisimple case

S_4:

```
----------------------
Class |   1  2  3  4  5
Size  |   1  3  6  8  6
Order |   1  2  2  3  4
----------------------
p  =  2   1  1  1  4  2
p  =  3   1  2  3  1  5
----------------------
X.1   +   1  1  1  1  1
X.2   +   1  1 -1  1 -1
X.3   +   2  2  0 -1  0
X.4   +   3 -1 -1  0  1
X.5   +   3 -1  1  0 -1
```

e.g. $\chi_5^2 = (3,-1,1,0,-1)^2 = (9,1,1,0,1) \leftrightsquigarrow [1,0,1,1,1]$

- **Character ring** $[\mathbf{Rep}(G)]$ = elements of the form $\sum_i c_i\chi_i$ for $c_i \in \mathbb{C}$ and multiplication = multiplication of characters
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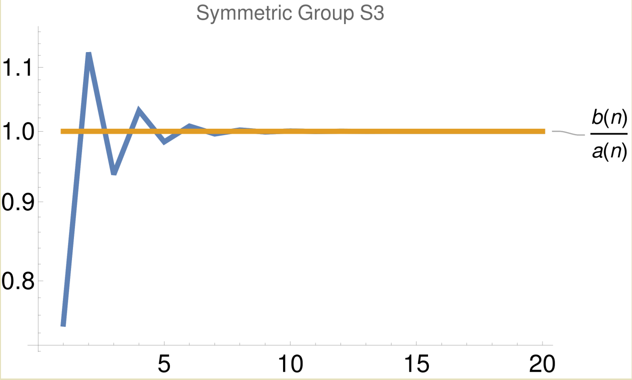
- **Example** $[\mathbf{Rep}(S_4)]$, χ_1 = unit $= 1 \leftrightsquigarrow$ trivial rep rest above

Finite groups - semisimple case

Asymptotic for S_4 and V = 3d standard rep

$$b_n \sim a_n = \frac{10}{24} \cdot n^0 \cdot 3^n$$



- Charac… $\in \mathbb{C}$ and multipli…
- Crucial … characters
- Example $[\mathbf{Rep}(S_4)]$, $\chi_1 = \text{unit} = 1 \leftrightsquigarrow$ trivial rep rest above

Finite groups - semisimple case

Asymptotic for G and V = any rep with $a > |b_i|$ for $[a, b_1, b_2, \ldots]$

$$b_n \sim a_n = \frac{\sum_{\text{simples}} \dim_{\mathbb{C}} L}{\sum_{\text{simples}} (\dim_{\mathbb{C}} L)^2} \cdot n^0 \cdot (\dim_{\mathbb{C}} V)^n$$

Class |		1	2	3	4	5	6	7
Size |		1	10	15	20	30	24	20
Order |		1	2	2	3	4	5	6
p =	2	1	1	1	4	3	6	4
p =	3	1	2	3	1	5	6	2
p =	5	1	2	3	4	5	1	7
X.1	+	1	1	1	1	1	1	1
X.2	+	1	-1	1	1	-1	1	-1
X.3	+	4	-2	0	1	0	-1	1
X.4	+	4	2	0	1	0	-1	-1
X.5	+	5	1	1	-1	-1	0	1
X.6	+	5	-1	1	-1	1	0	-1
X.7	+	6	0	-2	0	0	1	0

- Cha ... C and multi ...
- Cruc ... characters
- Example $[\mathbf{Rep}(S_4)]$, $\chi_1 = \text{unit} = 1 \leftrightsquigarrow$ trivial rep rest above

Finite groups - semisimple case

D_4:

```
----------------------
Class |   1  2  3  4  5
Size  |   1  1  2  2  2
Order |   1  2  2  2  4
----------------------
p  =  2   1  1  1  1  2
----------------------
X.1   +   1  1  1  1  1
X.2   +   1  1 -1  1 -1
X.3   +   1  1  1 -1 -1
X.4   +   1  1 -1 -1  1
X.5   +   2 -2  0  0  0
```

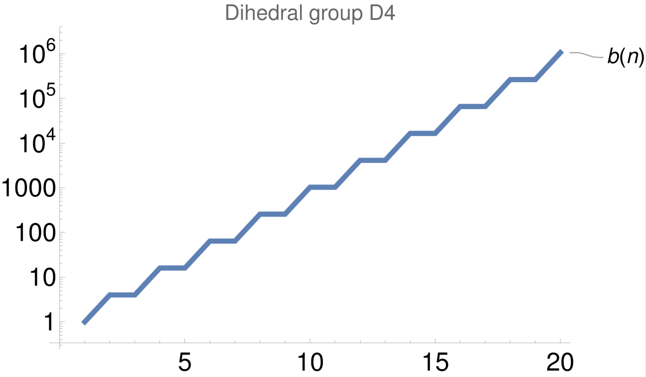
e.g. $\chi_5^2 = (2,-2,0,0,0)^2 = (4,4,0,0,0) \leftrightsquigarrow [1,1,1,1,0]$

- Character ring $[\mathbf{Rep}(G)]$ = elements of the form $\sum_i c_i\chi_i$ for $c_i \in \mathbb{C}$ and multiplication = multiplication of characters
- Crucial fact One can reconstruct $\mathbf{Rep}(G) = \mathbf{Rep}(G,\mathbb{C})$ from its characters
- Example $[\mathbf{Rep}(D_4)]$, $\chi_1 = \text{unit} = 1 \leftrightsquigarrow$ trivial rep rest above

Finite groups - semisimple case

Asymptotic for D_4 and V = 2d simple rep

$$b_n \sim a_n = \left(\tfrac{6}{8} + \tfrac{2}{8}(-1)^n\right) \cdot n^0 \cdot 2^n$$



- Chara… $\mathbb{C}$ and multipl…
- Crucial fact One can reconstruct $\mathbf{Rep}(G) = \mathbf{Rep}(G, \mathbb{C})$ from its characters
- Example $[\mathbf{Rep}(D_4)]$, $\chi_1 = \text{unit} = 1 \leftrightsquigarrow$ trivial rep rest above

Finite groups - semisimple case

$(\mathbb{Z}/3\mathbb{Z})^2 \rtimes (\mathbb{Z}/3\mathbb{Z})$:

Class		1	2	3	4	5	6	7	8	9	10	11
Size		1	1	1	3	3	3	3	3	3	3	3
Order		1	3	3	3	3	3	3	3	3	3	3
p =	3	1	1	1	1	1	1	1	1	1	1	1
X.1	+	1	1	1	1	1	1	1	1	1	1	1
X.2	0	1	1	1	J	-1-J	1	-1-J	1	J	J	-1-J
X.3	0	1	1	1	-1-J	1	J	J	-1-J	1	J	-1-J
X.4	0	1	1	1	-1-J	-1-J	-1-J	J	J	J	1	1
X.5	0	1	1	1	J	1	-1-J	-1-J	J	1	-1-J	J
X.6	0	1	1	1	1	J	-1-J	1	J	-1-J	J	-1-J
X.7	0	1	1	1	J	J	J	-1-J	-1-J	-1-J	1	1
X.8	0	1	1	1	1	-1-J	J	1	-1-J	J	-1-J	J
X.9	0	1	1	1	-1-J	J	1	J	1	-1-J	-1-J	J
X.10	0	3	3*J	-3-3*J	0	0	0	0	0	0	0	0
X.11	0	3	-3-3*J	3*J	0	0	0	0	0	0	0	0

$$J = \exp(2\pi i/3)$$

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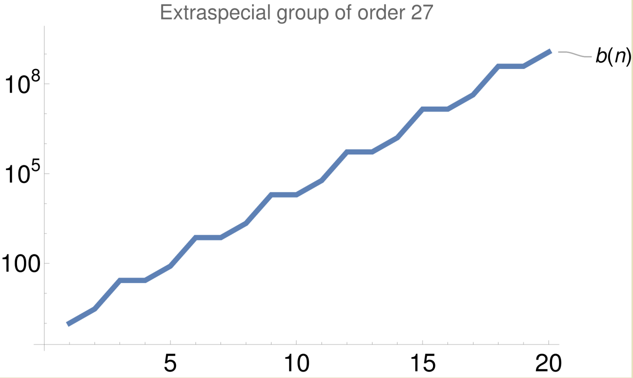
- Example $[\mathbf{Rep}((\mathbb{Z}/3\mathbb{Z})^2 \rtimes (\mathbb{Z}/3\mathbb{Z}))]$, $\chi_1 = \text{unit} = 1 \leftrightsquigarrow$ trivial rep rest above

Finite groups - semisimple case

Asymptotic for $(\mathbb{Z}/3\mathbb{Z})^2 \rtimes (\mathbb{Z}/3\mathbb{Z})$ and $V = $ 3d simple rep

$$b_n \sim a_n = \left(\tfrac{15}{27} + \tfrac{6}{27}J^n + \tfrac{6}{27}J^{2n}\right) \cdot n^0 \cdot 3^n$$



$(\mathbb{Z}/$

- Charac ... $\in \mathbb{C}$ and multipli...
- Crucial ... characters
- Example $[\mathbf{Rep}((\mathbb{Z}/3\mathbb{Z})^2 \rtimes (\mathbb{Z}/3\mathbb{Z}))]$, $\chi_1 = \text{unit} = 1 \rightsquigarrow$ trivial rep rest above

Finite groups - semisimple case

Class | 1 2 3 4 5 6 7 8 9 10 11

Asymptotic for G and V = any rep with $a \geq |b_i|, a \neq b_i$ and $a = |b_i|$ h-times for $[a, b_1, b_2, \ldots]$

$$b_n \sim a_n = \left(\frac{\text{column sum 1}}{|G|} + \frac{\text{column sum 2}}{|G|} J^n + \ldots\right) \cdot n^0 \cdot (\dim_{\mathbb{C}} V)^n \ (J{=}h \text{ root of unity})$$

Class		1	2	3	4	5	6	7	8	9	10	11
Size		1	1	1	3	3	3	3	3	3	3	3
Order		1	3	3	3	3	3	3	3	3	3	3
p =	3	1	1	1	1	1	1	1	1	1	1	1
X.1	+	1	1	1	1	1	1	1	1	1	1	1
X.2	0	1	1	1	J	-1-J	1	-1-J	1	J	J	-1-J
X.3	0	1	1	1	-1-J	1	J	J	-1-J	1	J	-1-J
X.4	0	1	1	1	-1-J	-1-J	-1-J	J	J	J	1	1
X.5	0	1	1	1	J	1	-1-J	-1-J	J	1	-1-J	J
X.6	0	1	1	1	1	J	-1-J	1	J	-1-J	J	-1-J
X.7	0	1	1	1	J	J	J	-1-J	-1-J	-1-J	1	1
X.8	0	1	1	1	1	-1-J	J	1	-1-J	J	-1-J	J
X.9	0	1	1	1	-1-J	J	1	J	1	-1-J	-1-J	J
X.10	0	3	3*J	-3-3*J	0	0	0	0	0	0	0	0
X.11	0	3	-3-3*J	3*J	0	0	0	0	0	0	0	0

- Example $[\mathbf{Rep}((\mathbb{Z}/3\mathbb{Z})^2 \rtimes (\mathbb{Z}/3\mathbb{Z}))]$, $\chi_1 = \text{unit} = 1 \rightsquigarrow$ trivial rep rest above

Finite groups - semisimple case

```
-------------------------------------------------------------
Class |   1     2     3    4    5    6    7    8    9   10   11
Size  |   1     1     1    3    3    3    3    3    3    3    3
```

$(\mathbb{Z}/3$

Asymptotic for G and V = any rep with 'otherwise'

Rep not faithful, find kernel and repeat with smaller group

```
------------------------
Class |   1  2  3  4  5
Size  |   1  3  6  8  6
Order |   1  2  2  3  4
------------------------
p  =  2   1  1  1  4  2
p  =  3   1  2  3  1  5
------------------------
X.1    +  1  1  1  1  1
X.2    +  1  1 -1  1 -1
X.3    +  2  2  0 -1  0
X.4    +  3 -1 -1  0  1
X.5    +  3 -1  1  0 -1
```

- Characte … $\in \mathbb{C}$ and multiplic…
- Crucial f … ts characters
- Example $[\mathbf{Rep}((\mathbb{Z}/3\mathbb{Z})^2 \rtimes (\mathbb{Z}/3\mathbb{Z}))]$, $\chi_1 = \text{unit} = 1 \rightsquigarrow$ trivial rep rest above

Finite groups - semisimple case

$\mathbb{Z}/2\mathbb{Z} \times S_4$:

		1	2	3	4	5	6	7	8	9	10
Class |		1	2	3	4	5	6	7	8	9	10
Size |		1	1	3	3	6	6	8	6	6	8
Order |		1	2	2	2	2	2	3	4	4	6
p =	2	1	1	1	1	1	1	7	3	3	7
p =	3	1	2	3	4	5	6	1	8	9	2
X.1	+	1	1	1	1	1	1	1	1	1	1
X.2	+	1	-1	1	-1	1	-1	1	-1	1	-1
X.3	+	1	1	1	1	-1	-1	1	-1	-1	1
X.4	+	1	-1	1	-1	-1	1	1	1	-1	-1
X.5	+	2	-2	2	-2	0	0	-1	0	0	1
X.6	+	2	2	2	2	0	0	-1	0	0	-1
X.7 ×	+	3	3	-1	-1	1	1	0	-1	-1	0
X.8	+	3	-3	-1	1	-1	1	0	-1	1	0
X.9 ×	+	3	-3	-1	1	1	-1	0	1	-1	0
X.10	+	3	3	-1	-1	-1	-1	0	1	1	0

- **Question** What is the growth of b_n for the marked reps?
- **Answer**

Finite groups - semisimple case

$\mathbb{Z}/2\mathbb{Z} \times S_4$:

Class		1	2	3	4	5	6	7	8	9	10
Size		1	1	3	3	6	6	8	6	6	8
Order		1	2	2	2	2	2	3	4	4	6
p = 2		1	1	1	1	1	1	7	3	3	7
p = 3		1	2	3	4	5	6	1	8	9	2
X.1	+	1	1	1	1	1	1	1	1	1	1
X.2	+	1	-1	1	-1	1	-1	1	-1	1	-1
X.3	+	1	1	1	1	-1	-1	1	-1	-1	1
X.4	+	1	-1	1	-1	-1	1	1	1	-1	-1
X.5	+	2	-2	2	-2	0	0	-1	0	0	1
X.6	+	2	2	2	2	0	0	-1	0	0	-1
X.7	+	3	3	-1	-1	1	1	0	-1	-1	0
X.8	+	3	-3	-1	1	-1	1	0	-1	1	0
X.9	+	3	-3	-1	1	1	-1	0	1	-1	0
X.10	+	3	3	-1	-1	-1	-1	0	1	1	0

- **Question** What is the growth of b_n for the marked reps?
- **Answer**

$$b_n \sim a_n = \left(\frac{20}{48} + \frac{0}{48}(-1)^n\right) \cdot n^0 \cdot 3^n$$

$$b_n \sim a_n = \frac{10}{24} \cdot n^0 \cdot 3^n$$

Finite groups - semisimple case

Recall Riemann hypothesis $\leftrightsquigarrow$ variance $|b_n - a_n|$

Here: $|b_n - a_n| \leq |\lambda_{sec}|^n$ for the second largest row entry λ_{sec}

```
-----------------------------------
Class |   1  2  3  4  5  6  7
Size  |   1 10 15 20 30 24 20
Order |   1  2  2  3  4  5  6
-----------------------------------
p  =  2   1  1  1  4  3  6  4
p  =  3   1  2  3  1  5  6  2
p  =  5   1  2  3  4  5  1  7
-----------------------------------
X.1   +   1  1  1  1  1  1  1
X.2   +   1 -1  1  1 -1  1 -1
X.3   +   4 -2  0  1  0 -1  1
X.4   +   4  2  0  1  0 -1 -1
X.5   +   5  1  1 -1 -1  0  1
X.6   +   5 -1  1 -1  1  0 -1
X.7   +   6  0 -2  0  0  1  0
```

Example b_n grows like 6^n and $|b_n - a_n|$ is bounded by 2^n

- Questi
- Answe

$b_n \sim a_n = (\frac{20}{48} + \frac{0}{48}(-1)^n) \cdot n^0 \cdot 3^n$

$b_n \sim a_n = \frac{10}{24} \cdot n^0 \cdot 3^n$

Finite groups - nonsemisimple case

$\mathbb{Z}/5\mathbb{Z}$ over $\bar{\mathbb{F}}_5$:

$$\mathrm{Z}_1 = \mathrm{L}_1 : \begin{pmatrix} \boxed{1} \end{pmatrix}, \textit{is simple}$$

$$\mathrm{Z}_2 : \begin{pmatrix} \boxed{1} & 0 \\ \boxed{1} & \boxed{1} \end{pmatrix}, \textit{filtration } 0 - \mathrm{L}_1 - \mathrm{Z}_2,$$

$$\mathrm{Z}_3 : \begin{pmatrix} \boxed{1} & 0 & 0 \\ \boxed{1} & \boxed{1} & 0 \\ 0 & \boxed{1} & \boxed{1} \end{pmatrix}, \textit{filtration } 0 - \mathrm{L}_1 - \mathrm{L}_1 - \mathrm{Z}_3,$$

$$\mathrm{Z}_4 : \begin{pmatrix} \boxed{1} & 0 & 0 & 0 \\ \boxed{1} & \boxed{1} & 0 & 0 \\ 0 & \boxed{1} & \boxed{1} & 0 \\ 0 & 0 & \boxed{1} & \boxed{1} \end{pmatrix}, \textit{filtration } 0 - \mathrm{L}_1 - \mathrm{L}_1 - \mathrm{L}_1 - \mathrm{Z}_4,$$

$$\mathrm{Z}_5 = \mathrm{P}_1 : \begin{pmatrix} \boxed{1} & 0 & 0 & 0 & 0 \\ \boxed{1} & \boxed{1} & 0 & 0 & 0 \\ 0 & \boxed{1} & \boxed{1} & 0 & 0 \\ 0 & 0 & \boxed{1} & \boxed{1} & 0 \\ 0 & 0 & 0 & \boxed{1} & \boxed{1} \end{pmatrix}, \textit{filtration } 0 - \mathrm{L}_1 - \mathrm{L}_1 - \mathrm{L}_1 - \mathrm{L}_1 - \mathrm{Z}_5,$$

- $\mathbb{Z}/5\mathbb{Z}$ over $\bar{\mathbb{F}}_5$ = five indecomposables $\rightsquigarrow$ five Jordan blocks
- Dimensions are $1, 2, 3, 4, 5$
- Z_1 is simple, Z_5 is projective

Finite groups - nonsemisimple

b_n for Z_1

Boring: $b_n = 1^n$
since $\dim_{\bar{\mathbb{F}}_5} Z_1 = 1$

$\mathbb{Z}/5\mathbb{Z}$ over $\bar{\mathbb{F}}_5$:

$$\mathrm{Z}_1 = \mathrm{L}_1 : \left(1\right)$$

$$\mathrm{Z}_2 : \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \textit{filtration } 0 - \mathrm{L}_1 - \mathrm{Z}_2,$$

$$\mathrm{Z}_3 : \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \textit{filtration } 0 - \mathrm{L}_1 - \mathrm{L}_1 - \mathrm{Z}_3,$$

$$\mathrm{Z}_4 : \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}, \textit{filtration } 0 - \mathrm{L}_1 - \mathrm{L}_1 - \mathrm{L}_1 - \mathrm{Z}_4,$$

$$\mathrm{Z}_5 = \mathrm{P}_1 : \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}, \textit{filtration } 0 - \mathrm{L}_1 - \mathrm{L}_1 - \mathrm{L}_1 - \mathrm{L}_1 - \mathrm{Z}_5,$$

- $\mathbb{Z}/5\mathbb{Z}$ over $\bar{\mathbb{F}}_5$ = five indecomposables $\rightsquigarrow$ five Jordan blocks
- Dimensions are $1, 2, 3, 4, 5$
- Z_1 is simple, Z_5 is projective

Finite groups - nonsemisimple

b_n for Z_1

Boring: $b_n = 1^n$
since $\dim_{\bar{\mathbb{F}}_5} Z_1 = 1$

b_n for Z_5

$$b_n = \tfrac{1}{5} \cdot n^0 \cdot 5^n$$
$$Z_5^2 = Z_5 \otimes Z_5 \cong Z_5^{\oplus 5} = 5Z_5$$

$\mathrm{Z}_1 = \mathrm{L}_1 :$

$-\mathrm{Z}_3,$

$\mathbb{Z}/5\mathbb{Z}$ over $\bar{\mathbb{F}}_5$:

$$\mathrm{Z}_4 : \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}, \textit{filtration } 0 - \mathrm{L}_1 - \mathrm{L}_1 - \mathrm{L}_1 - \mathrm{Z}_4,$$

$$\mathrm{Z}_5 = \mathrm{P}_1 : \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}, \textit{filtration } 0 - \mathrm{L}_1 - \mathrm{L}_1 - \mathrm{L}_1 - \mathrm{L}_1 - \mathrm{Z}_5,$$

- $\mathbb{Z}/5\mathbb{Z}$ over $\bar{\mathbb{F}}_5$ = five indecomposables $\leftrightsquigarrow$ five Jordan blocks
- Dimensions are $1, 2, 3, 4, 5$
- Z_1 is simple, Z_5 is projective

b_n for Z_1

Boring: $b_n = 1^n$
since $\dim_{\overline{\mathbb{F}}_5} Z_1 = 1$

b_n for Z_5

$$b_n = \frac{1}{5} \cdot n^0 \cdot 5^n$$
$$Z_5^2 = Z_5 \otimes Z_5 \cong Z_5^{\oplus 5} = 5Z_5$$

b_n for Z_3 (Z_2, Z_4 analog)

$$b_n \sim a_n = \frac{1}{5} \cdot n^0 \cdot 3^n$$
since $Z_3 \otimes Z_3 \cong Z_1 \oplus Z_3 \oplus Z_5$ and $Z_3 \otimes Z_5 \cong 3Z_5$

$$Z_3 \leftrightsquigarrow [0,1,0]$$
$$Z_3^2 \leftrightsquigarrow [1,1,1]$$
$$Z_3^3 \leftrightsquigarrow [1,2,4]$$
$$Z_3^4 \leftrightsquigarrow [2,3,14]$$
$$Z_3^5 \leftrightsquigarrow [3,5,45]$$
$$Z_3^6 \leftrightsquigarrow [5,8,140]$$

$$(b_n)_{n\in\mathbb{N}} = (1,1,3,7,19,53,153,\dots)$$
$$(a_n)_{n\in\mathbb{N}} = (0.2,0.6,1.8,5.4,16.2,48.6,145.8,\dots)$$

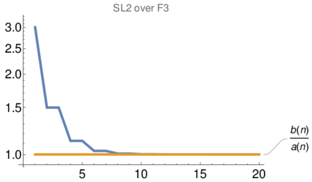
$Z_1 = L_1 :$ ... $- Z_3,$ ... $- L_1 - Z_5,$

$\mathbb{Z}/5\mathbb{Z}$ over $\overline{\mathbb{F}}$...

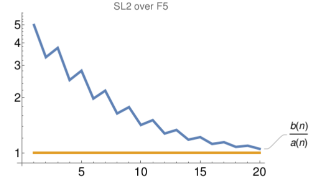
- $\mathbb{Z}/5\mathbb{Z}$ over $\overline{\mathbb{F}}$... ...cks
- Dimensions ...
- Z_1 is simple ...

Finite groups - nonsemisimple case

$p = 3$:



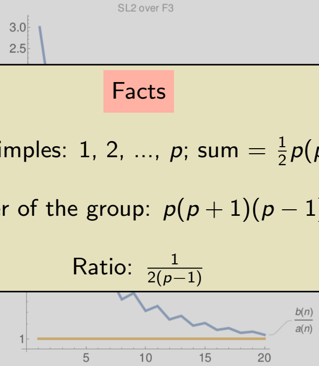
$p = 5$:



- Example $\mathrm{SL}_2(\mathbb{F}_p)$, $\mathbb{K} = \mathbb{F}_p$ and $V = \mathbb{F}_p^2$
- Growth

$$b_n \sim a_n = \frac{1}{2(p-1)}\left(1 + \frac{1}{p}(-1)^n\right) \cdot n^0 \cdot 2^n$$

Finite groups - nonsemisimple case



Facts

Dims of simples: 1, 2, ..., p; sum $= \frac{1}{2}p(p+1)$

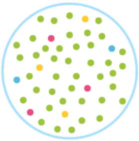
Order of the group: $p(p+1)(p-1)$

Ratio: $\frac{1}{2(p-1)}$

- Example $\mathrm{SL}_2(\mathbb{F}_p)$, $\mathbb{K} = \mathbb{F}_p$ and $V = \mathbb{F}_p^2$
- Growth

$$b_n \sim a_n = \frac{1}{2(p-1)}\left(1 + \frac{1}{p}(-1)^n\right) \cdot n^0 \cdot 2^n$$

Finite groups - nonsemisimple case



- Theorem (very inclusive) For finite groups and b_n, there is essentially no difference between char 0 and char p
- Precisely $\mathbb{K} = \bar{\mathbb{K}}$, V our rep

$$b_n \sim a_n = \frac{\sum_{\text{simples}} \dim_{\mathbb{K}} L}{|G|}(1 + xx) \cdot n^0 \cdot (\dim_{\mathbb{K}} V)^n$$

with $xx = c_1 J^n + c_2 J^{2n} + \ldots + c_h J^{hn}$ for $J = \exp(2\pi i/h)$

- There is also a version for the variance

Finite groups - nonsemisimple case

S_4 over $\bar{\mathbb{F}}_3$:

```
----------------------
Class |   1  2  3  4  5
Size  |   1  3  6  8  6
Order |   1  2  2  3  4
----------------------
p  =  2   1  1  1  4  2
p  =  3   1  2  3  1  5
----------------------
X.1   +   1  1  1  1  1
X.2   +   1  1 -1  1 -1
X.3   +   2  2  0 -1  0
X.4   +   3 -1 -1  0  1
X.5   +   3 -1  1  0 -1
```

+

- Question What is the growth of b_n for the χ_5 rep?
- Answer

char 0:

char 3

Finite groups - nonsemisimple case

S_4 over $\bar{\mathbb{F}}_3$:

```
------------------------
Class |   1  2  3  4  5
Size  |   1  3  6  8  6
Order |   1  2  2  3  4
------------------------
p  =  2   1  1  1  4  2
p  =  3   1  2  3  1  5
------------------------
X.1   +   1  1  1  1  1
X.2   +   1  1 -1  1 -1  +
X.3   +   2  2  0 -1  0
X.4   +   3 -1 -1  0  1
X.5   +   3 -1  1  0 -1
```

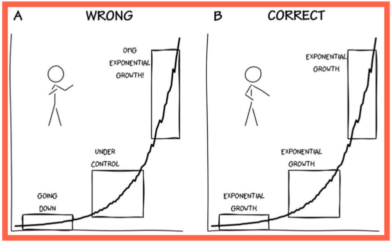
- **Question** What is the growth of b_n for the χ_5 rep?
- **Answer**

char 0: $b_n \sim a_n = \frac{8}{24} \cdot n^0 \cdot 3^n$ char 3: $b_n \sim a_n = \frac{10}{24} \cdot n^0 \cdot 3^n$

Finite groups - nonsemisimple case

Analytic theory of monoidal categories

Or: How to not count



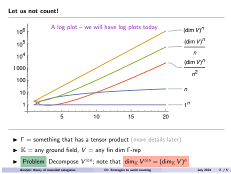
This is part 2

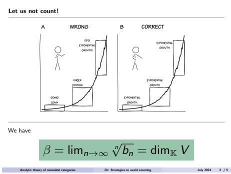
Theorem (very inclusive)

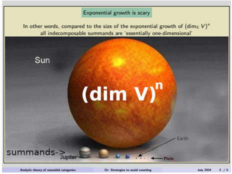
The same works for any finite tensor category
by replacing dim with PFdim

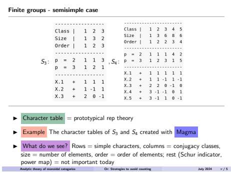
- Questio
- Answer

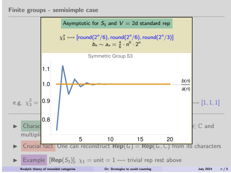
char 0: $b_n \sim a_n = \frac{8}{24} \cdot n^0 \cdot 3^n$ char 3: $b_n \sim a_n = \frac{10}{24} \cdot n^0 \cdot 3^n$

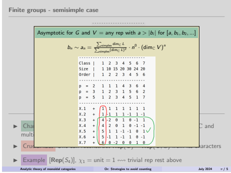


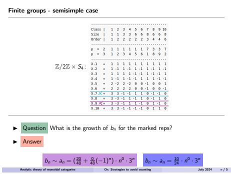


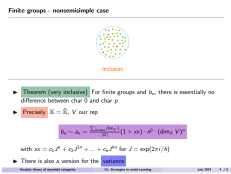


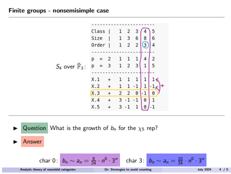




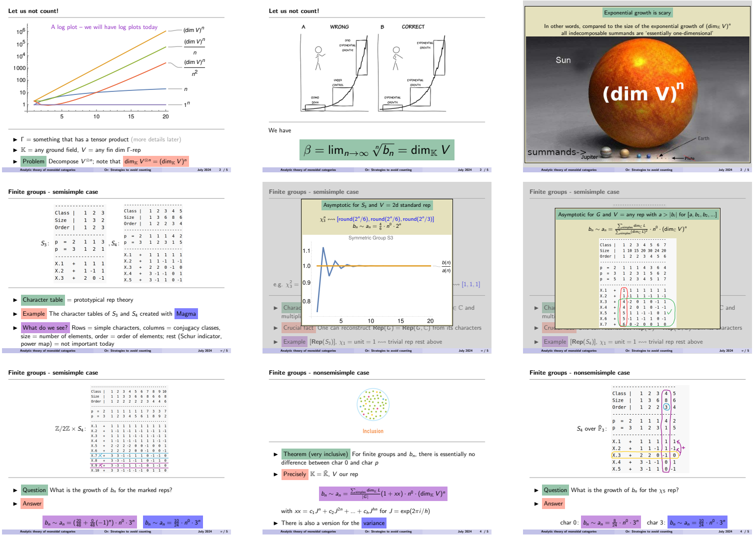








There is still much to do...



Thanks for your attention!