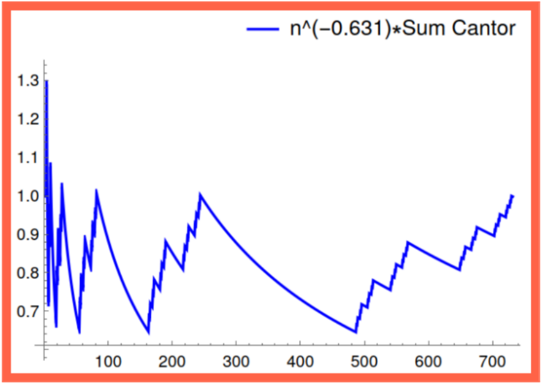
Analytic theory of monoidal categories

Or: Strategies to avoid counting



This is part 3

Reminder: How we do not count!

- Γ = some group
- $\mathbb{K}$ = any ground field, V = any fin dim Γ-rep
- Problem Find the growth rate of b_n=# inde. summands in $V^{\otimes n}$

Reminder: How we do not count!

Finite groups - semisimple case

$\mathbb{Z}/2\mathbb{Z} \times S_4$:

```
-----------------------------------
Class |   1  2  3  4  5  6  7  8  9 10
Size  |   1  1  3  3  6  6  8  6  6  8
Order |   1  2  2  2  2  2  3  4  4  6
-----------------------------------
p  =  2   1  1  1  1  1  1  7  3  3  7
p  =  3   1  2  3  4  5  6  1  8  9  2
-----------------------------------
X.1   +   1  1  1  1  1  1  1  1  1  1
X.2   +   1 -1  1 -1  1 -1  1 -1  1 -1
X.3   +   1  1  1  1 -1 -1  1 -1 -1  1
X.4   +   1 -1  1 -1 -1  1  1  1 -1 -1
X.5   +   2 -2  2 -2  0  0 -1  0  0  1
X.6   +   2  2  2  2  0  0 -1  0  0 -1
X.7 × +   3  3 -1 -1  1  1  0 -1 -1  0
X.8   +   3 -3 -1  1 -1  1  0 -1  1  0
X.9 × +   3 -3 -1  1  1 -1  0  1 -1  0
X.10  +   3  3 -1 -1 -1 -1  0  1  1  0
```

- Question What is the growth of b_n for the marked reps?
- Answer

$b_n \sim a_n = (\frac{20}{48} + \frac{0}{48}(-1)^n) \cdot n^0 \cdot 3^n$ $\quad$ $b_n \sim a_n = \frac{10}{24} \cdot n^0 \cdot 3^n$

Analytic theory of monoidal categories | Or: How to not count | July 2024 | π / 5

$b_n \sim a_n = h \cdot n^\tau \cdot \beta^n$

h=bounded function

n^τ subexponential factor

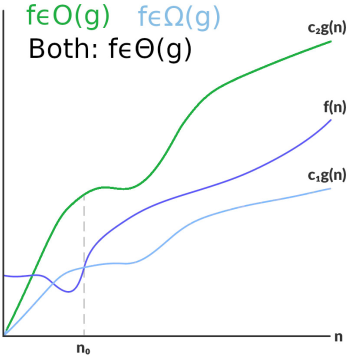
β^n exponential factor

- Reminder For finite groups we always have

h='scalar' | $\tau = 0$ trivial subexponential factor | $\beta = \dim_{\mathbb{K}} V$ exponential factor

- Question What happens for more general groups?

Reminder: How we do not count!



- **Theorem (subexp. factor)** For $\operatorname{char}\mathbb{K}=0$, the following are equivalent:
  (i) $b_n \in \Theta((\dim_{\mathbb{K}} V)^n)$
  (ii) The connected component of the Zariski closure of the image of Γ in $\mathrm{GL}(V)$ is a torus
- **Translation** Read (i) as $\tau = 0$; read (ii) as $\Gamma = \mathbb{Z}^m \rtimes$ (finite group)

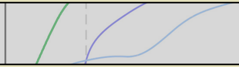
Reminder: How we do not count!

Three features to note

(1) The exponential factor only depends on the representation

(2) The subexponential factor does not depend on the representation

(3) The asymptotic only depends on the image of Γ in $\mathrm{GL}(V)$



Note that we studied "small" images of Γ in $\mathrm{GL}(V)$ (finite groups)

Next, we focus on $\Gamma = \mathrm{GL}(V)$ equiv. $\Gamma = \mathrm{SL}(V)$, say for $\mathbb{K} = \mathbb{C}$

$\dim_{\mathbb{C}} V = 1$ is boring, so we start with $\dim_{\mathbb{C}} V = 2$

$$\mathrm{SL}(2, \mathbb{C}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{C} \text{ and } ad - bc = 1 \right\}$$

- The ... lent:
  - (i)
  - (ii) The connected component of the Zariski closure of the image of Γ in $\mathrm{GL}(V)$ is a torus
- Translation Read (i) as $\tau = 0$; read (ii) as $\Gamma = \mathbb{Z}^m \rtimes$ (finite group)

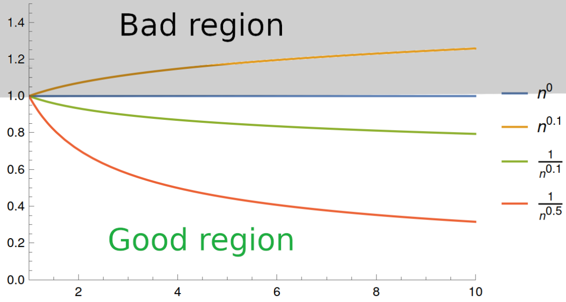
Reminder: How we do not count!

Recall

(1) The subexponential factor is always $\beta = \dim_{\mathbb{K}} V$

(2) We have $b_n \leq (\dim_{\mathbb{K}} V)^n$

(3) Thus, the τ in the subexponential factor satisfies $\tau \leq 0$



- Theorem ... uivalent:
  - (i) b_n ...
  - (ii) The ... $V)$ is a torus
- Translation Read (i) as $\tau = 0$; read (ii) as $\Gamma = \mathbb{Z}^m \rtimes$ (finite group)

Reminder: How we do not count!

$$b_n \sim h(n) \cdot n^\tau \cdot \beta^n$$

$h\colon \mathbb{Z}_{\geq 0} \to \mathbb{R}_{>0}$ is a function *bounded away from* $0, \infty$,
n^τ is the *subexponential factor*, $\tau \in \mathbb{R}$,
β^n is the *exponential factor*, $\beta \in \mathbb{R}_{\geq 1}$.

- Ansatz: What to expect from not counting
- Any sequence of numbers b_n counting something (not just in monoidal categories) often satisfies the above
- h is often a constant but sometimes h is more complicated

Reminder: How we do not count!

Examples

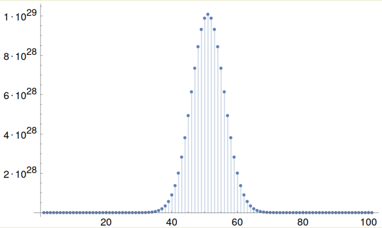
(1) (Prime counting function $\pi'(n) = \pi(e^n)$) $\sim 1 \cdot n^{-1} \cdot e^n$

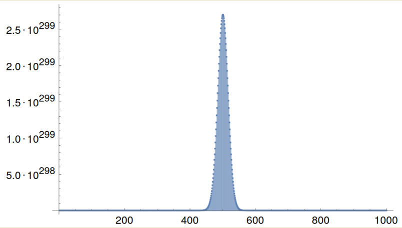
(2) (Number of partitions of n^2) $\sim 1/(4\sqrt{3}) \cdot n^{-2} \cdot (e^{\sqrt{2/3}\pi})^n$

(3) (Number of trees with n vertices) $\sim C \cdot n^{-5/2} \cdot D^n$ with $C \approx 0.535$, $D \approx 2.996$

(4) (Rabbit counting à la Fibonacci) $\sim \sqrt{5} \cdot 1 = n^0 \cdot \phi^n$

(5) (Middle binomials $\binom{n}{n/2}$) $\sim \sqrt{2/\pi} \cdot n^{-1/2} \cdot 2^n$



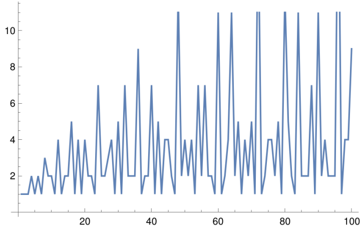


categories) often satisfies the above

- h is often a constant but sometimes h is more complicated

Reminder: How we do not count!

Examples (cont.)

#mult. partitions: 

- **Multiplicative partition** = an ordered way of writing an integer as a product of positive integers ≥ 2, e.g. $12 = 3 \cdot 2 \cdot 2$

Better to look at: the average $\sum_{k=1}^{n} m(k)/n$, $m(k)$=#mult. partitions

(Average #mult. partitions of e^{n^2}) $\sim$ $1/(2\pi)$ $\cdot$ $n^{-3/2}$ $\cdot$ $(e^2)^n$

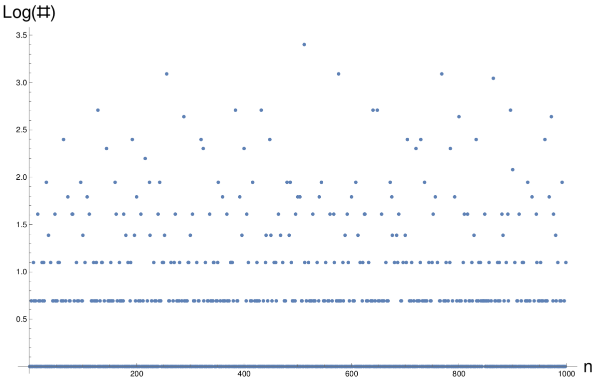
categories) often satisfies the above

- h is often a constant but sometimes h is more complicated

Examples (cont.) (This is a reminder)

A000688 Number of Abelian groups of order n; number of factorizations of n into prime powers. 129
(Formerly M0064 N0020)

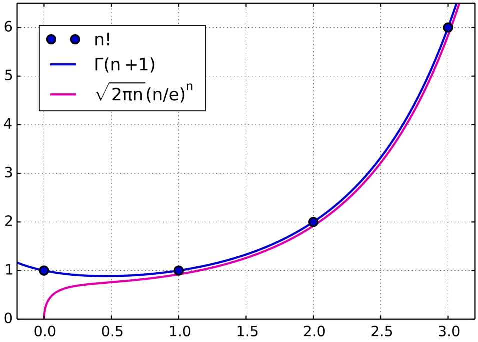
1, 1, 1, 2, 1, 1, 1, 3, 2, 1, 1, 2, 1, 1, 1, 5, 1, 2, 1, 2, 1, 1, 1, 3, 2, 1, 3, 2, 1, 1, 1, 7, 1,
1, 1, 4, 1, 1, 1, 3, 1, 1, 1, 2, 2, 1, 1, 5, 2, 2, 1, 2, 1, 3, 1, 3, 1, 1, 1, 2, 1, 1, 2, 11, 1,
1, 1, 2, 1, 1, 1, 6, 1, 1, 2, 2, 1, 1, 1, 5, 5, 1, 1, 2, 1, 1, 1, 3, 1, 2, 1, 2, 1, 1, 1, 7, 1, 2,
2, 4, 1, 1, 1, 3, 1, 1, 1 (list; graph; refs; listen; history; text; internal format)



The average $\sum_{k=1}^{n} agnu(k)/n$, $agnu(k)$=#abelian groups of order k

(Average #abelian groups of order n) $\sim$ $\prod_{j\geq 2}\zeta(j)$ $\cdot$ $1 = n^0$ $\cdot$ $1 = 1^n$, $\prod_{j\geq 2}\zeta(j) \approx 2.295$

Reminder: How we do not count!

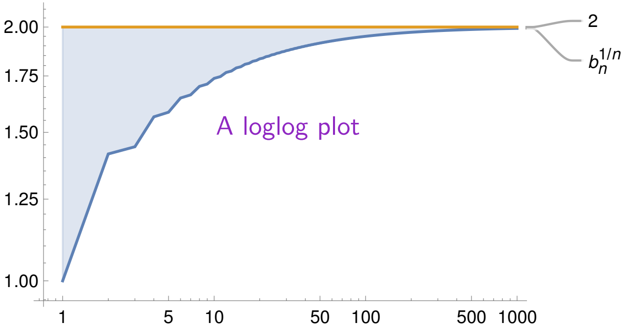


- Question How large (dim wise) do reps of S_n get?
- Compare $|S_n| = n! \sim$ above and $(\#$ partitions of $n) \sim 1/(4\sqrt{3}) \cdot n^{-1} \cdot (e^{\sqrt{2/3}\pi})^{\sqrt{n}}$
- Hence, one should expect that S_n has large reps and indeed $e^{-n}(n/e)^{n/2} < \max\dim(S_n\text{-reps})$

General groups - characteristic zero

- Γ = a group ($SL_2(\mathbb{C})$ most of the time)
- $\mathbb{K} = \mathbb{C}$, V = any fin dim Γ-rep
- Problem Decompose $V^{\otimes n}$ - too difficult, better: count summands

General groups - characteristic zero

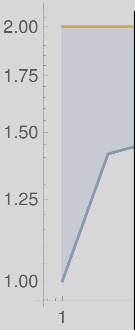


- $b_n = b_n^{\Gamma,V}$=number of indecomposable summands of $V^{\otimes n}$ (with multiplicities)
- **Example** $\Gamma = SL_2$, $\mathbb{K} = \mathbb{C}$, $V = \mathbb{C}^2$ (vector rep), then

$$\{1, 1, 2, 3, 6, 10, 20, 35, 70, 126, 252\}, \quad b_n \text{ for } n = 0, ..., 10.$$

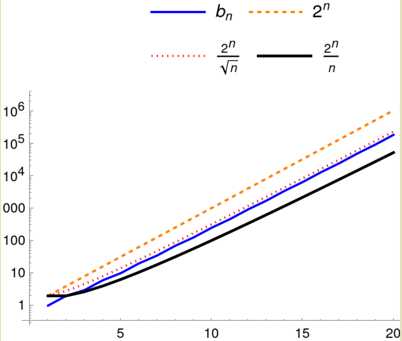
Research task Copy the sequence and put it into OEIS

General groups - characteristic zero



b_n = middle binomials Hence:

$$b_n \sim \sqrt{2/\pi} \cdot n^{-1/2} \cdot 2^n$$



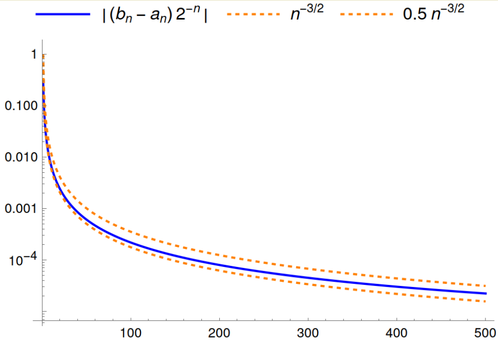
- $b_n = b_n^{\Gamma,V}$=numbe ... $^{\otimes n}$ (with multiplicities)
- **Example** $\Gamma = SL_2$ ...

$$\{1, 1, 2, 3, 6, 10, 20, 35, 70, 126, 252\}, \quad b_n \text{ for } n = 0, ..., 10.$$

Research task Copy the sequence and put it into OEIS

Even the variance is doable :

$$|b_n - a_n| \sim C \cdot n^{-3/2} \cdot 2^n$$

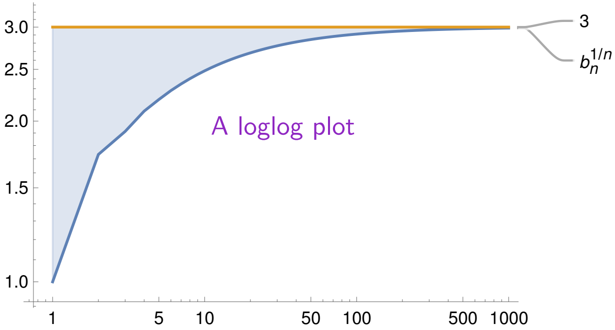


Proof? Scroll through the OEIS page for the middle binomials

- $b_n = b$ … multiplicities)
- **Examp** … 0.

Research task Copy the sequence and put it into OEIS

General groups - characteristic zero



- $b_n = b_n^{\Gamma,V}$=number of indecomposable summands of $V^{\otimes n}$ (with multiplicities)
- **Example** $\Gamma = SL_2$, $\mathbb{K} = \mathbb{C}$, $V = Sym^2\mathbb{C}^2$ (the 3d simple), then

$$\{1, 1, 3, 7, 19, 51, 141, 393, 1107, 3139, 8953\}, \quad b_n \text{ for } n = 0, ..., 10.$$

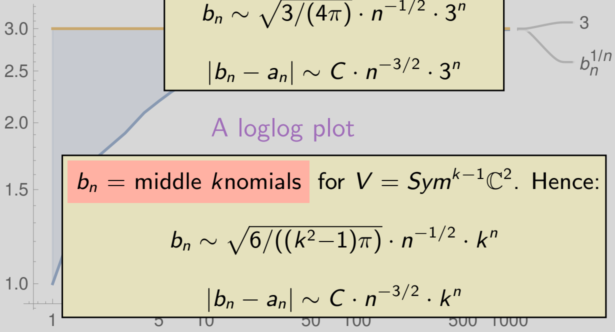
Research task Copy the sequence and put it into OEIS

General groups - charac

b_n = middle trinomials Hence:

$$b_n \sim \sqrt{3/(4\pi)} \cdot n^{-1/2} \cdot 3^n$$

$$|b_n - a_n| \sim C \cdot n^{-3/2} \cdot 3^n$$



b_n = middle $k$nomials for $V = Sym^{k-1}\mathbb{C}^2$. Hence:

$$b_n \sim \sqrt{6/((k^2-1)\pi)} \cdot n^{-1/2} \cdot k^n$$

$$|b_n - a_n| \sim C \cdot n^{-3/2} \cdot k^n$$

- $b_n = b_n^{\Gamma,V}$=num ... (with multiplicities)
- **Example** $\Gamma = S$ ... , then

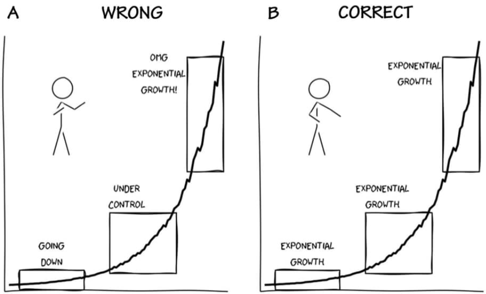
$\{1, 1, 3, 7,$ ... or $n = 0, ..., 10$.

Conjecture (for b_n)

Dominating growth is $\beta = \dim_{\mathbb{K}} V$
Subexponential factor n^{τ} only depends on Γ
h is a scalar

Research task Copy the sequence and put it into OEIS

General groups - characteristic zero



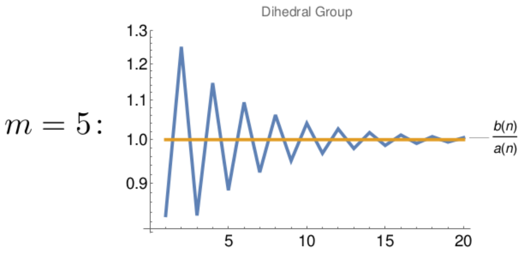
- Theorem A The dominating growth is always the dimension (proven for all semigroup superschemes Γ, all fields, all fd reps V)
- Theorem B n^{τ} only depends on Γ (proven for all groups, characteristic zero fields, all fd reps V)
- Theorem C h takes only finitely many values (proven for all groups, characteristic zero fields, all fd reps V)

General group

Example

Dihedral group of order 10, $\mathbb{K}=\mathbb{C}$, $V=$any simple 2d

$$b_n \sim \left(\tfrac{7}{10}+\tfrac{1}{5}(-1)^n\right)\cdot n^0\cdot 2^n$$



Comments (char zero)

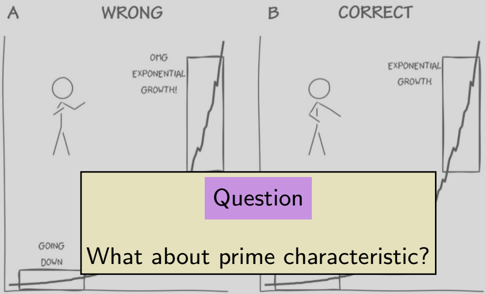
For Γ simple reductive group, $\tau=-\#$pos. roots/2
and $h=$scalar given by closed formula, variance=closed formula

For Γ finite group, $\tau=0$
and $h+$variance computable from the character table

- Theor… …ven for all semigr…
- Theor… …eristic zero fields, …
- …

The above is due to many people, e.g.:
Biane, Bryant–Kovács, Coulembier–Etingof–Ostrik, Lacabanne–Vaz, He,...

General groups - characteristic zero



Question

What about prime characteristic?

- Theorem A The dominating growth is always the dimension (proven for all semigroup superschemes Γ, all fields, all fd reps V)
- Theorem B n^{τ} only depends on Γ (proven for all groups, characteristic zero fields, all fd reps V)
- Theorem C h takes only finitely many values (proven for all groups, characteristic zero fields, all fd reps V)

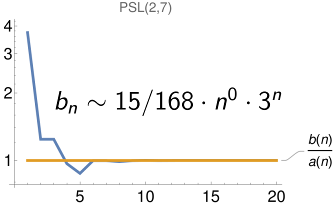
SL2 - prime characteristic

$PSL_2(\mathbb{F}_7)$

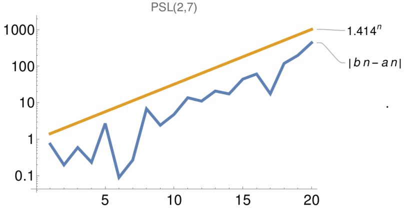
$\mathbb{K} = \mathbb{F}_2$:

V=any 3d simple

$b_n \sim 15/168 \cdot n^0 \cdot 3^n$



Variance:

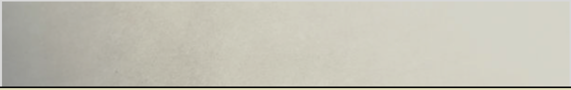


- Γ = a finite group, $\mathbb{K}$ = any ground field, V = any fin dim Γ-rep
- **Coulembier–Etingof–Ostrik, Lacabanne–Vaz, He ∼2024** This works as in char zero

SL2 - prime characteristic

- Done char zero: all groups; char p: finite groups
- Next $\Gamma = SL_2(\bar{\mathbb{F}}_p)$, $\mathbb{K} = \bar{\mathbb{F}}_p$
- We will see a remarkable complexity jump

SL2 - prime characteristic



One finds fractals in asymptotic counting problems in monoidal categories defined over fields of prime characteristic

Next Two non-monoidal primers due to:

Haboush ∼1980 (first)

Carter–Cline ∼1976 (second)

Coulembier–Etingof–Ostrik ∼2024 (put together)

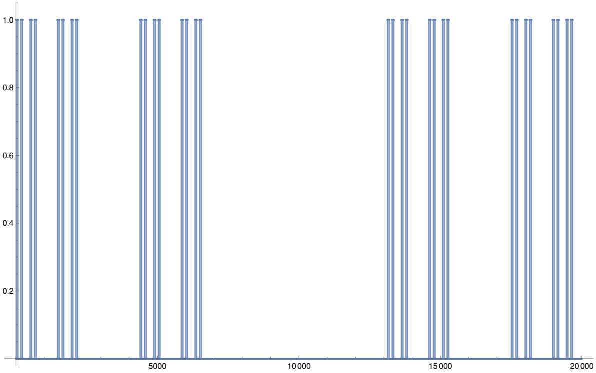
After that The monoidal case due to:

Larsen, Coulembier–Etingof–Ostrik ∼2024

- Done char zero: all groups; char p: finite groups
- Next $\Gamma = SL_2(\bar{\mathbb{F}}_p)$, $\mathbb{K} = \bar{\mathbb{F}}_p$
- We will see a remarkable complexity jump

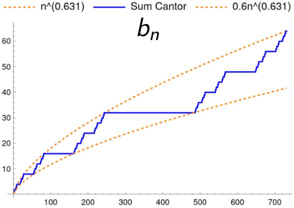
SL2 - prime characteristic

ca_n:



- For $p = 3$, let $L_{-1/2}$ be the simple rep of highest weight $-1/2$
- ca_n = the dimension of its weight space of weight $-1/2 - n$
- $b_n = \sum_{k=0}^{n} ca_k$, which quantifies the growth of $L_{-1/2}$ satisfies $h(n) \cdot n^{\tau} \cdot \beta^n$ with $\beta = 1$ Recall: if you see the above, take the sum

SL2 - prime characteristic



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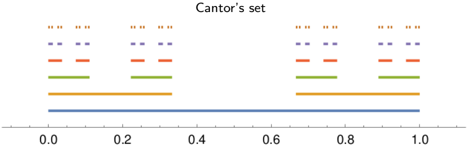

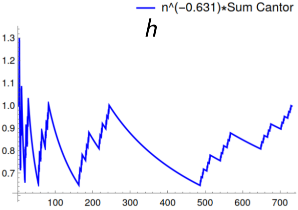

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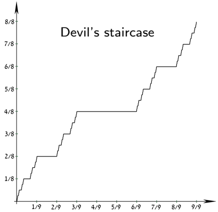
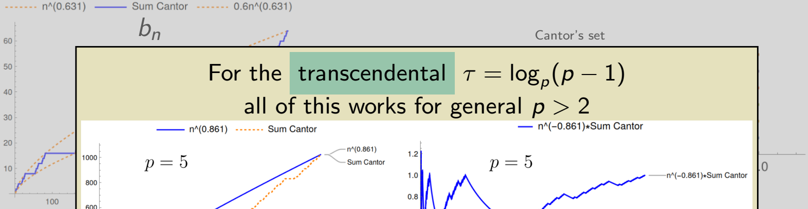


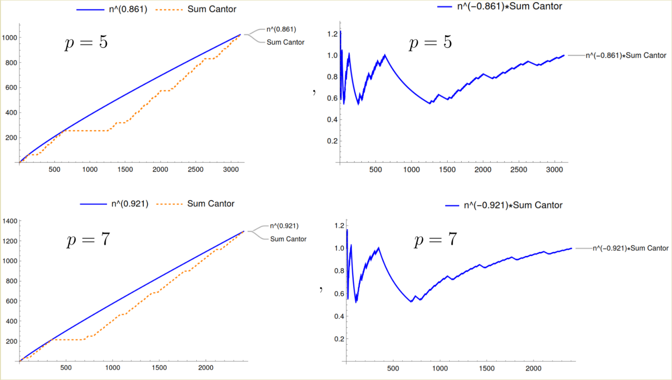
- New 1 $\tau = \log_3 2 = \text{dim of Cantor set} \approx 0.631$
- New 2 h is insane: it approaches a periodic function akin to devil's staircase

SL2 - prime characteristic



For the transcendental $\tau = \log_p(p-1)$

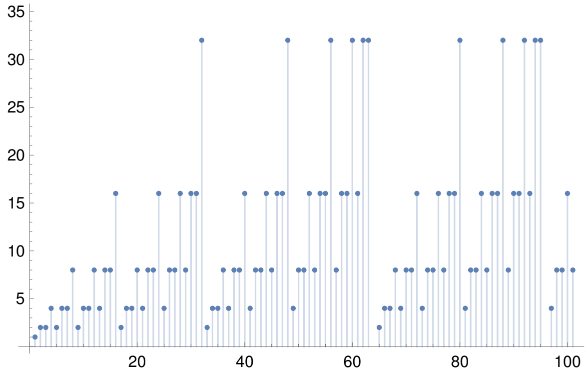
all of this works for general $p > 2$



- New 1 $\tau = \log_3 2 =$ dim of Cantor set ≈ 0.631
- New 2 h is insane: it approaches a periodic function akin to devil's staircase

SL2 - prime characteristic

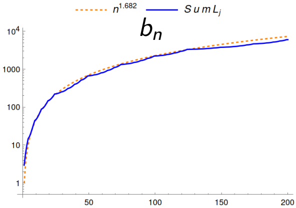
$\dim L_n$:



- For $p = 2$, let L_n be the simple rep of highest weight $n \in \mathbb{N}$
- $\dim L_n$ = the dimension of it
- $b_n = \sum_{k=0}^{n} \dim L_k$, which quantifies the growth of L_n satisfies $h(n) \cdot n^{\tau} \cdot \beta^{n}$ with $\beta = 1$ Recall: if you see the above, take the sum

SL2 - prime characteristic

$p = 5$:



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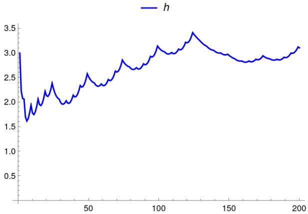
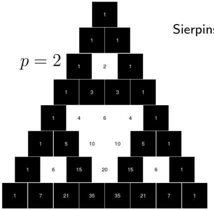


$p = 2$



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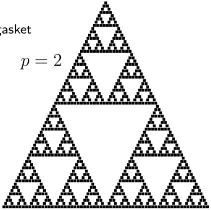
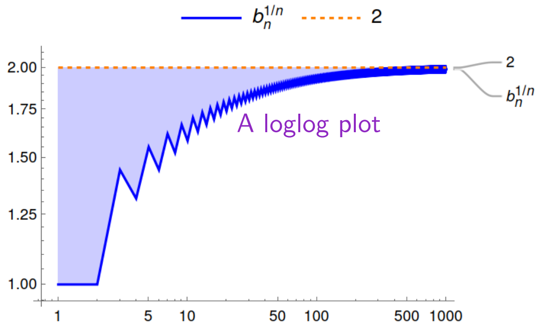
Sierpinski's gasket

$p = 2$



- **New 1** $\tau = 1 + \log_p \frac{p+1}{2}$ = dim of Sierpinski's gasket ≈ 1.682 for $p = 5$
- **New 2** h is again insane

SL2 - prime characteristic



- $b_n = b_n^{\Gamma,V}$=number of indecomposable summands of $V^{\otimes n}$ (with multiplicities)
- **Example** $\Gamma = SL_2$, $\mathbb{K} = \bar{\mathbb{F}}_p$, $V = \bar{\mathbb{F}}_p^2$ (vector rep), then

$$\{1, 1, 1, 3, 3, 9, 9, 29, 29, 99, 99\}, \quad b_n \text{ for } n = 0, ..., 10.$$

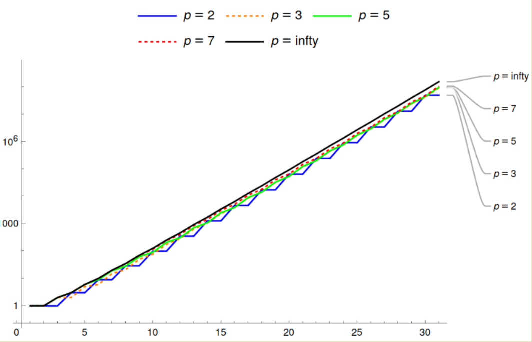
Research task Copy the sequence and put it into OEIS

SL2 - prime characteristic

$b_n^{1/n}$ 2

b_n = new sequence

$$b_n \sim h(n) \cdot n^{1/2 \log_p \frac{p+1}{2} - 1} \cdot 2^n$$



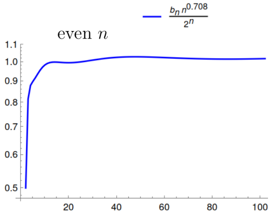
- $b_n = b_n^{\Gamma, V} =$ … ith multiplicities)
- **Example Γ** …

$\{1, 1, 1, 3, 3, 9, 9, 29, 29, 99, 99\}$, b_n for $n = 0, ..., 10$.

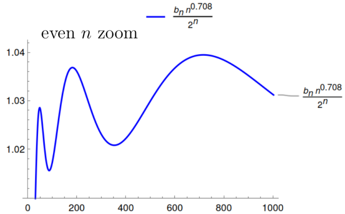
Research task Copy the sequence and put it into OEIS

SL2 - prime characteristic

$h, p = 2$:



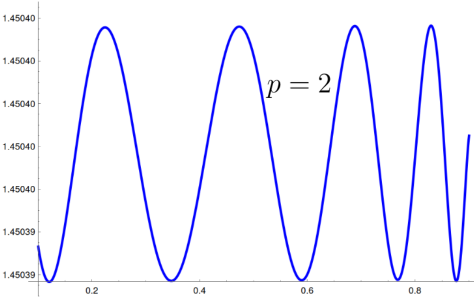
$h, p = 2$:



- New 1 $\tau = 1/2 \log_p \frac{p+1}{2} - 1 = \text{dim of ???} \approx -0.708$ for $p = 2$
- New 2 h is again insane

SL2 - prime characteristic

zoom in $h, p = 2$:



- *h* is really insane It has ∞ many nonzero Fourier coefficients L_n (highly oscillating)
- Some analytic number theory going on:
  - The L_n involve the (Hurwitz) zeta and Gamma function
  - There are functional equations akin to Mahler functions and Dirichlet's L-function

SL2 - prime ch

$$b_n \sim h(n) \cdot n^{\tau} \cdot \beta^n$$

Recall the char zero results:

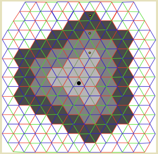
- Theorem A The dominating growth is always the dimension (proven for all semigroup superschemes Γ, all fields, all fd reps V)
- Theorem B n^{τ} only depends on Γ (proven for all groups, characteristic zero fields, all fd reps V)
- Theorem C h takes only finitely many values (proven for all groups, characteristic zero fields, all fd reps V)

Wannabe theorem 1 in prime characteristic

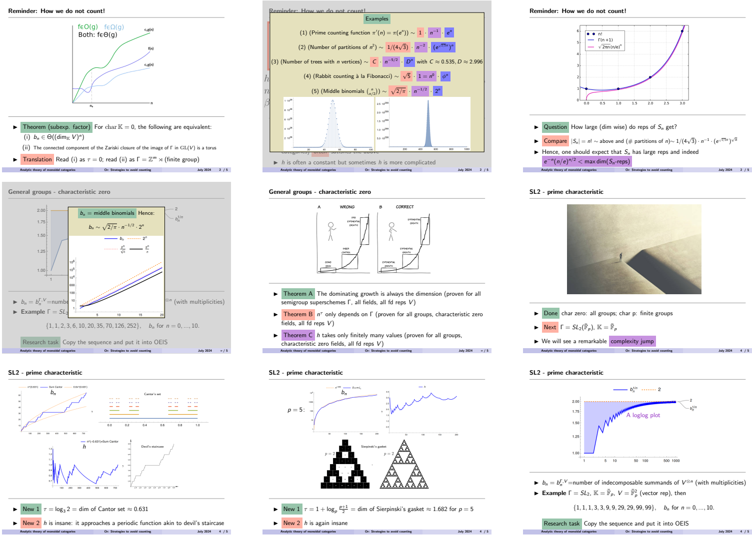
Theorem B also holds (Theorem A is always true, Theorem C is false)

Wannabe theorem 2 (Last talk)

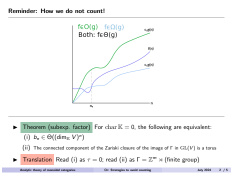
Something similar works for the Hecke category

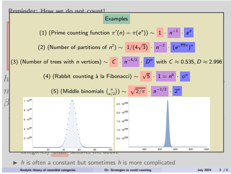


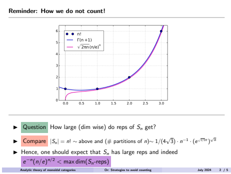
- h is really ins … nts L_n (highly oscillating)
- Some analyti …
  - The L_n i … on
  - There are … ns and Dirichlet's L-functio …

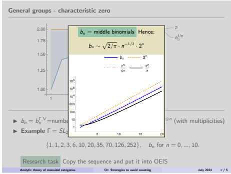


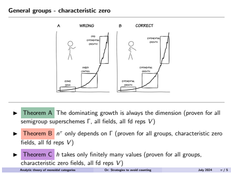
There is still much to do...

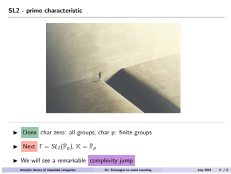


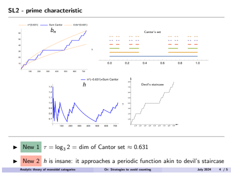


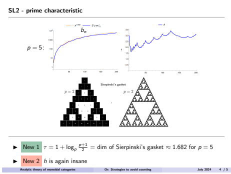


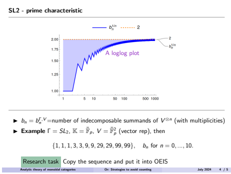












Thanks for your attention!