Analytic theory of monoidal categories

Or: Strategies to avoid counting

This is part 4

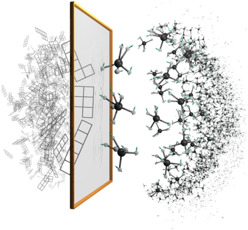
Let us not count!

$$U_q(sl_n) \looparrowright V_q^{\otimes n} \looparrowleft H(S_n)$$

Quantum Groups — Hecke Algebra

$$SL(V) \looparrowright V^{\otimes n} \looparrowleft S_n$$

Special Linear Group — Symmetric Group



- Schur–Weyl duality relates $\mathrm{SL}_\beta/\mathrm{GL}_\beta$-reps and S_n-reps in $V^{\otimes n}$ for $V = \mathbb{C}^\beta$
- So far We have studied $b_n{=}\#$ inde. summands of $V^{\otimes n}$
- Task Study the Schur–Weyl dual of b_n

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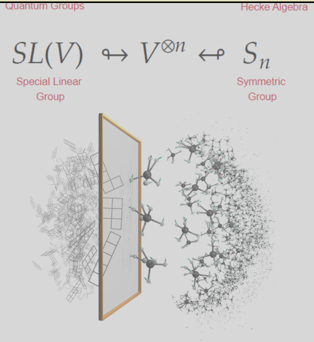
Theorem (Schur–Weyl dual of b_n)

A acts on $V^{\otimes n}$, $b_n = \#$ inde. summands of $V^{\otimes n}$, $B = \mathrm{End}_A(V^{\otimes n})$
Then b_n=sum of dims of simple B-reps



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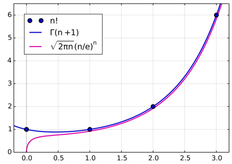
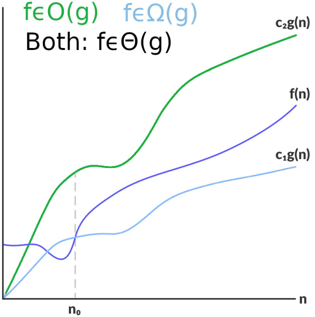
Quantum Groups Hecke Algebra

Reminder: How we do not count!



- **Question** How large (dim wise) do reps of S_n get?
- **Compare** $|S_n| = n! \sim$ above and (# partitions of n)$\sim 1/(4\sqrt{3}) \cdot n^{-1} \cdot (e^{\sqrt{2/3}\pi})^{\sqrt{n}}$
- Hence, one should expect that S_n has large reps and indeed $e^{-n}(n/e)^{n/2} < \max\dim(S_n\text{-reps})$



Example

Total sum of dims of S_n-reps is in $\Omega(\beta^n)$ for all $\beta \in \mathbb{R}_{>0}$

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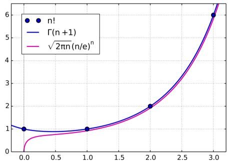

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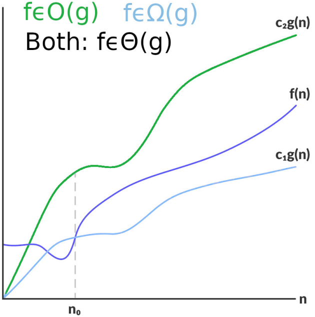

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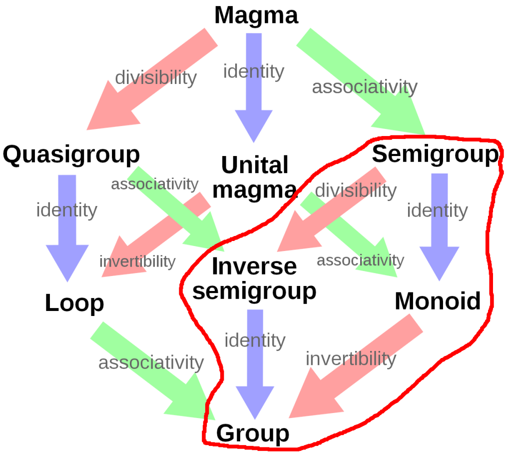
$\mathrm{End}_A(V^{\otimes n})$ is a monoid

This motivates the study of **growth/bounds** of dimensions of monoid reps

Let us not count!



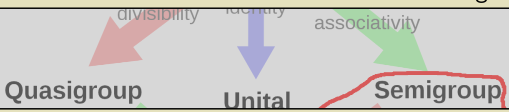
- Associativity ⇒ reasonable theory of matrix reps
- Southeast corner ⇒ reasonable theory of matrix reps

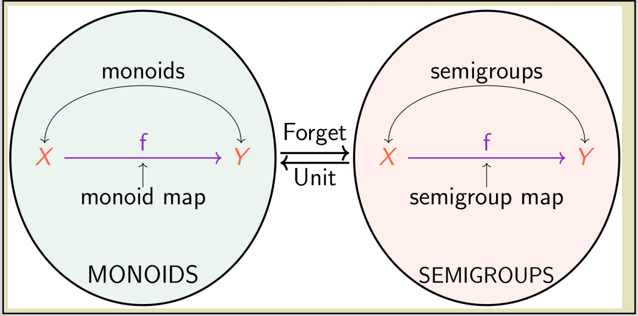
Let us

Adjoining identities is "free" and there is no essential difference between semigroups and monoids

The main difference is monoids vs. groups

I will stick with the more familiar monoids and groups





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divisibility identity associativity

In a monoid information is destroyed

The point of monoid theory is to keep track of information loss

identity identity

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The point of monoid theory is to keep track of information loss

Monoids appear naturally in categorification

Group-like structures

	Totality$^{\alpha}$	Associativity	Identity	Invertibility	Commutativity
Semigroupoid	Unneeded	Required	Unneeded	Unneeded	Unneeded
Small category	Unneeded	Required	Required	Unneeded	Unneeded
Groupoid	Unneeded	Required	Required	Required	Unneeded
Magma	Required	Unneeded	Unneeded	Unneeded	Unneeded
Quasigroup	Required	Unneeded	Unneeded	Required	Unneeded
Unital magma	Required	Unneeded	Required	Unneeded	Unneeded
Semigroup	Required	Required	Unneeded	Unneeded	Unneeded
Loop	Required	Unneeded	Required	Required	Unneeded
Inverse semigroup	Required	Required	Unneeded	Required	Unneeded
Monoid	Required	Required	Required	Unneeded	Unneeded
Commutative monoid	Required	Required	Required	Unneeded	Required
Group	Required	Required	Required	Required	Unneeded
Abelian group	Required	Required	Required	Required	Required

- Associativity =
- Southeast corr

Let us not count!

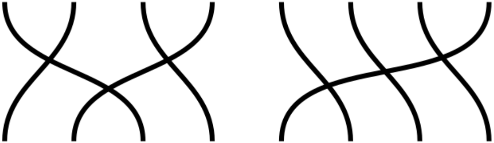
Examples of monoids

Groups

Multiplicative closed sets of matrices (these need not to be unital, but anyway)

Symmetric groups $\mathrm{Aut}(\{1,...,n\})$

$(24138567) \leftrightsquigarrow$



Transformation monoids $\mathrm{End}(\{1,...,n\})$

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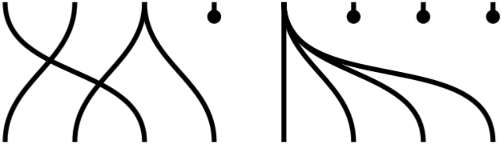

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Let us not count!

Example

$\mathbb{Z}$ is a group Integers

$\mathbb{N}$ is a monoid Natural numbers

Quasigroup Unital Semigroup

Example

$C_n = \langle a | a^n = 1 \rangle$ is a group Cyclic group

$C_{n,p} = \langle a | a^{n+p} = a^n \rangle$ is a monoid Cyclic monoid

identity

Example (now with notation)

$\mathfrak{S}_n = \mathrm{Aut}(\{1, ..., n\})$ is a group Symmetric group

$\mathfrak{T}_n = \mathrm{End}(\{1, ..., n\})$ is a monoid Transformation monoid

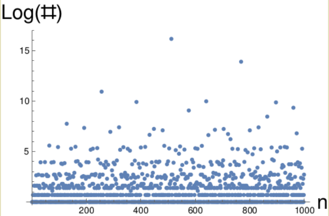
- Associativity $\Rightarrow$ reasonable theory of matrix reps
- Southeast corner $\Rightarrow$ reasonable theory of matrix reps

Let us

Finite groups are kind of random...

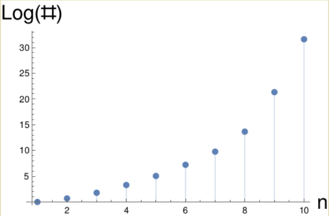
A000001 Number of groups of order n.
(Formerly M0098 N0035)

0, 1, 1, 1, 2, 1, 2, 1, 5, 2, 2, 1, 5, 1, 2, 1, 14, 1, 5, 1, 5, 2, 2, 1, 15, 2, 2, 5, 4, 1, 4, 1, 51, 1, 2, 1, 14, 1, 2, 2, 14, 1, 6, 1, 4, 2, 2, 1, 52, 2, 5, 1, 5, 1, 15, 2, 13, 2, 2, 1, 13, 1, 2, 4, 267, 1, 4, 1, 5, 1, 4, 1, 50, 1, 2, 3, 4, 1, 6, 1, 52, 15, 2, 1, 15, 1, 2, 1, 12, 1, 10, 1,



A058133 Number of monoids (semigroups with identity) of order n, considered to be equivalent when they are isomorphic or anti-isomorphic (by reversal of the operator).

0, 1, 2, 6, 27, 156, 1373, 17730, 858977, 1844075697, 52991253973742 (list; graph; refs; listen; history;



▶ A

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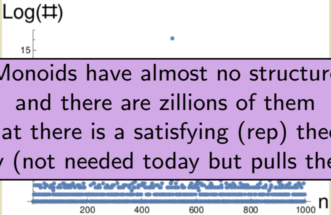

Monoids have almost no structure
and there are zillions of them
⇒ not clear that there is a satisfying (rep) theory of monoids
There is: Green's cell theory (not needed today but pulls the strings in the background)

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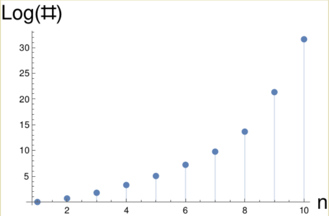

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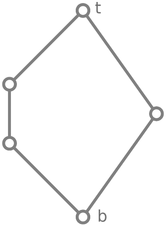
Representation gap



- S = monoid, $G \subset S$ = group of units
- S has two trivial reps, called bottom and top:

$$1_b \colon S \to \mathbb{K}, \quad s \mapsto \begin{cases} 1 & \text{if } s \in G, \\ 0 & \text{else}, \end{cases} \quad 1_t \colon S \to \mathbb{K}, \quad s \mapsto 1.$$

- The name comes from the fact that simple monoid reps are partially ordered and these are at bottom/top

Representation gap

Example
S is a group
$\Leftrightarrow$
$S = G$
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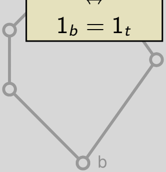
$1_b = 1_t$



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$S = S_{0;1} = \langle a | a^2 = a \rangle$ is essentially trivial

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Example (monoid 2 with two elements)

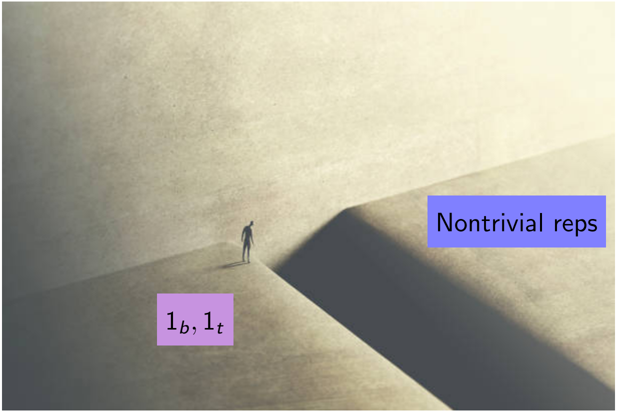
$S = \langle a | a^2 = 1 \rangle$ (this is $\mathbb{Z}/2\mathbb{Z}$)

$\Rightarrow$

$1_b = 1_t$ and $a \mapsto -1$ are the only simple S-reps

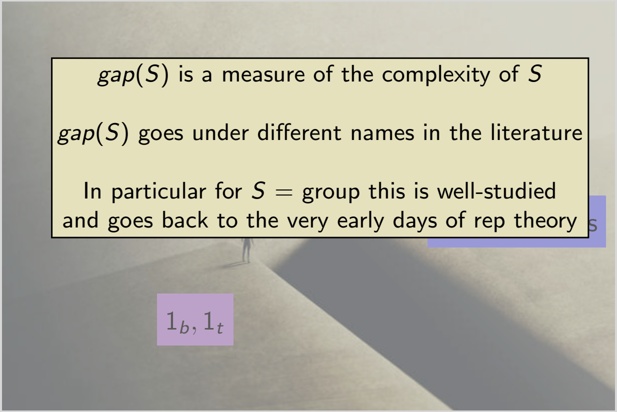
- The name co… partially ordered and these are …

Representation gap



- Call all S-reps $1_t^{\oplus m} \oplus 1_b^{\oplus n}$ trivial
- Rep gap $gap_{\mathbb{K}}(S)$ = smallest dim of a nontrivial S-rep over $\mathbb{K}$; $gap_* = \min$ of $gap_{\mathbb{K}}(S)$ over all fields $\mathbb{K}$; write $gap(S)$ if the difference doesn't matter

Representation gap



gap(S) is a measure of the complexity of S

gap(S) goes under different names in the literature

In particular for $S =$ group this is well-studied and goes back to the very early days of rep theory

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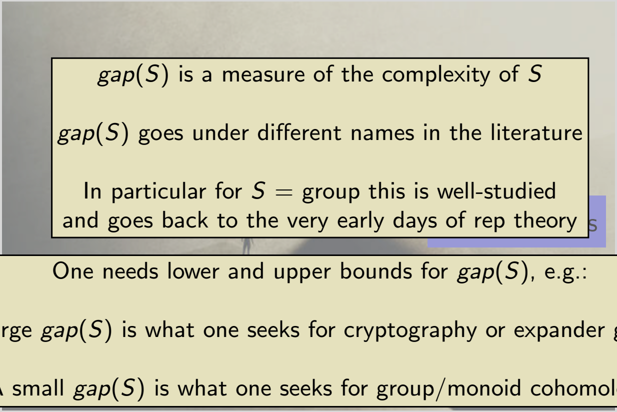

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One needs lower and upper bounds for *gap*(S), e.g.:

A large *gap*(S) is what one seeks for cryptography or expander graphs

A small *gap*(S) is what one seeks for group/monoid cohomology

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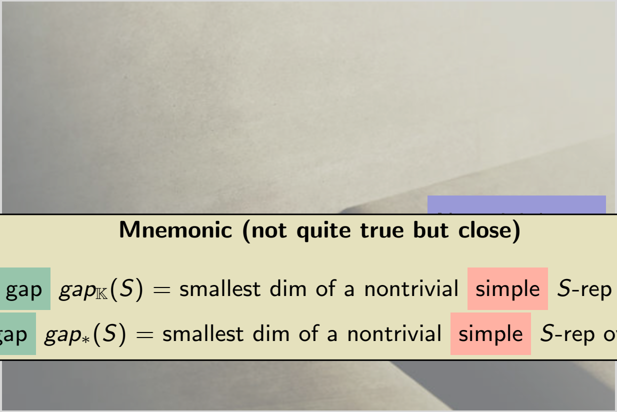


Mnemonic (not quite true but close)

Rep gap $gap_{\mathbb{K}}(S)$ = smallest dim of a nontrivial simple S-rep over $\mathbb{K}$

Rep gap $gap_{*}(S)$ = smallest dim of a nontrivial simple S-rep over all $\mathbb{K}$

- Call all S-reps $1_t^{\oplus m} \oplus 1_b^{\oplus n}$ trivial
- Rep gap $gap_{\mathbb{K}}(S)$ = smallest dim of a nontrivial S-rep over $\mathbb{K}$; gap_{*} = min of $gap_{\mathbb{K}}(S)$ over all fields $\mathbb{K}$; write $gap(S)$ if the difference doesn't matter

Representation gap

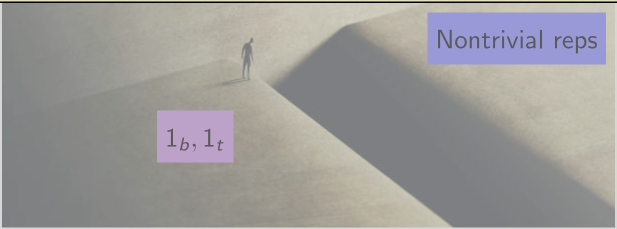
Example/convention

For $S = \{1\}$ we define $gap(S) = 0$

For $S = S_{0;1} = \{0, 1\}$ we define $gap(S) = 0$

Why? These are the only monoids without any nontrivial reps

so $gap(S)$ would be infinite, but that is silly...



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Example (groups)

For $S = \mathbb{Z}/2\mathbb{Z}$ we have $gap_{\mathbb{C}}(S) = 1$

For $S = \mathfrak{S}_n = Aut(\{1, ..., n\})$ we have $gap_{\mathbb{C}}(S) = 1$

For $S =$Monster we have $gap_{\mathbb{C}}(S) = 196883$ **(Griess ∼1980 and others)**

For $S = SL_2(\mathbb{F}_p)$ we have $gap_{\mathbb{C}}(S) \geq \frac{p-1}{2}$ **(Frobenius ∼1900)**

For $S = \mathbb{Z}/2\mathbb{Z}$ we have $gap_{*}(S) = 1$

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For $S =$Monster we have $gap_{*}(S) \leq 196882$ **(Griess–Smith ∼1994)**

For $S = SL_2(\mathbb{F}_p)$ we have $gap_{*}(S) = 2$ since we can act on $\mathbb{F}_p^2$

- Ca
- R $=$ min of $gap_{\mathbb{K}}(S)$ over all fields $\mathbb{K}$; write $gap(S)$ if the difference doesn't matter

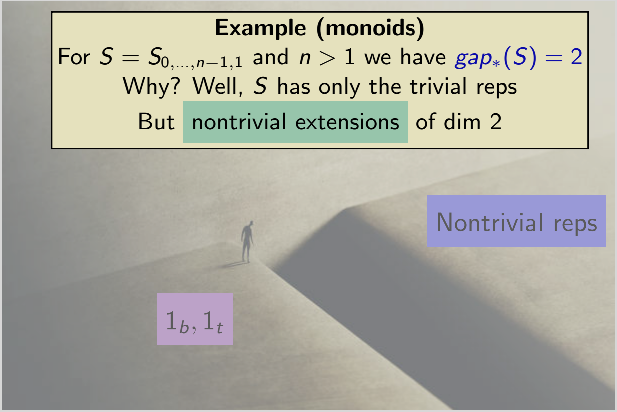
Representation gap

Example (monoids)

For $S = S_{0,\dots,n-1,1}$ and $n > 1$ we have $gap_*(S) = 2$

Why? Well, S has only the trivial reps

But nontrivial extensions of dim 2



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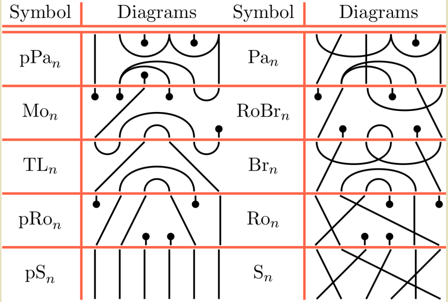
Example (monoids)

There will be some results for diagram monoids momentarily



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R

Honorable mentions

Old Faithful

Eruption of Old Faithful in 1948

Alternatively, and studied in group theory since the early days (under various names) and by e.g. **Mazorchuk–Steinberg ∼2011** in monoid theory one could use faithfulness as a measure of complexity (using the same notation):

Faithfulness $faith(S)$ = smallest dim of a faithful rep

- Call all S-reps $1_t^{\oplus m} \oplus 1_b^{\oplus n}$ trivial
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Faithfulness *faith*(S) = smallest dim of a faithful rep

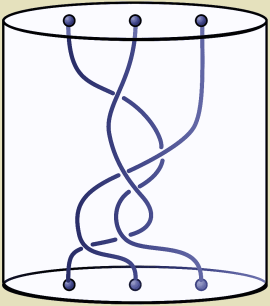
Examples

For $S = \mathfrak{S}_n = \mathrm{Aut}(\{1, \dots, n\})$ for $n \geq 5$ we have $faith_{\mathbb{C}}(S) = n - 1$ **(Burnside ∼1902)**
For $S = \mathfrak{T}_n = \mathrm{End}(\{1, \dots, n\})$ we have $faith_{\mathbb{C}}(S) = n$ **(Mazorchuk–Steinberg ∼2011)**

For $S = \mathfrak{S}_n = \mathrm{Aut}(\{1, \dots, n\})$ for $n \geq 5$ we have $faith_{*}(S) = n - 2$ **(Dickson ∼1908)**
For $S = \mathfrak{T}_n = \mathrm{End}(\{1, \dots, n\})$ we have $faith_{*}(S) = ??$

Theorem (easy)
Under some silly nontriviality assumptions on S:
$gap(S) \leq faith(S) \leq |S|$

Example (infinite group but still...)

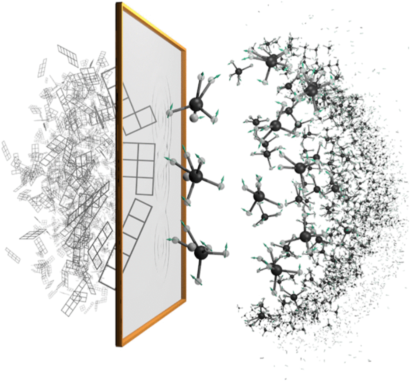


For the braid group Br_n on n strands we have
$gap_{\mathbb{Q}(q,t)}(Br_n) \leq n-1$, $faith_{\mathbb{Q}(q,t)}(Br_n) \leq n(n-1)/2$
dim Burau $= n-1$, dim LKB $= n(n-1)/2$

- Call all S-re
- Rep gap g ... r $\mathbb{K}$; $gap_* =$ min of $gap_{\mathbb{K}}(S)$ over all fields $\mathbb{K}$; write $gap(S)$ if the difference doesn't matter

Rep gap and monoidal categories

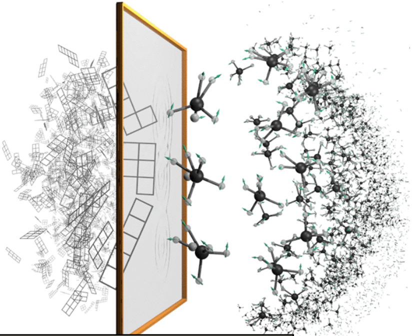
Schur–Weyl duality relates two objects



- For any monoidal category $\mathcal{C}$ we get a family of monoids $S_n = \mathrm{End}_{\mathcal{C}}(V^{\otimes n})$
- Schur–Weyl duality suggests that S_n should have a big rep gap
- Dim simple of S_n "$\Leftrightarrow$" # of indecomposables in $V^{\otimes n}$ and these grow fast

Rep gap and monoidal categories

Schur–Weyl duality relates two objects



Now: rep gap of the
"easiest Schur–Weyl duality monoid"

- For any monoidal category $\mathcal{C}$ we get a family of monoids $S_n = \mathrm{End}_{\mathcal{C}}(V^{\otimes n})$
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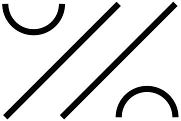
Rep gap and monoidal categories

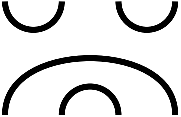
Connect 4 points at the bottom with 4 points at the top without crossings, potentially turning back:

$$\{\{1,-3\},\{2,-4\},\{3,4\},\{-1,-2\}\} \leftrightsquigarrow$$



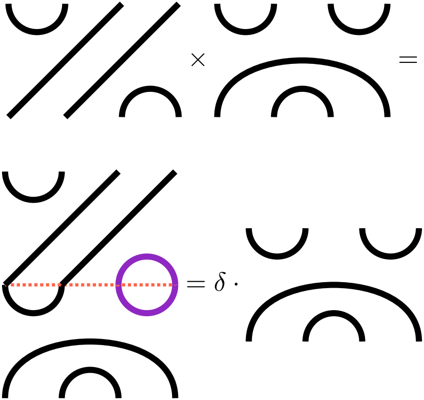
or

$$\{\{1,4\},\{2,3\},\{-1,-2\},\{-3,-4\}\} \leftrightsquigarrow$$



This is the Temperley–Lieb (TL) monoid TL_4 on $\{1,...,4\} \cup \{-1,...,-4\}$
In combinatorics, these are crossingless perfect matchings

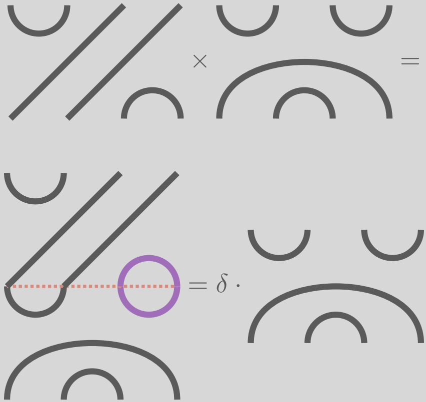
Rep gap and monoidal categories



Fix some field $\mathbb{K}$ and $\delta \in \mathbb{K}$, evaluate circles to $\delta \Rightarrow$ TL algebra $TL_4(\delta)$
The TL monoid is the non-linear version of $TL_4(1)$

Rep gap and m

The TL monoid TL_n arise (kicking out scalars) under Schur–Weyl duality as
$TL_n \cong \mathrm{End}_{U_q(gl_2)}\big((\mathbb{C}_q^2)^{\otimes n}\big)$ for $-q - q^{-1} = 1$



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The TL algebra goes back to **Rumer–Teller–Weyl** $\sim$**1932**

Zur Theorie der Spinvalenz.

Von

Georg Rumer in Moskau.

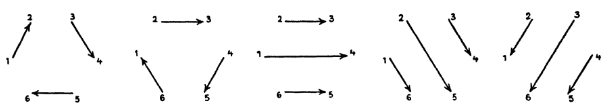
Vorgelegt von H. Weyl in der Sitzung am 22. Juli 1932.

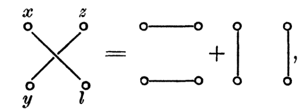
Eine für die Valenztheorie geeignete Basis der binären Vektorinvarianten.

Von

G. Rumer (Moskau), **E. Teller** und **H. Weyl** (Göttingen).

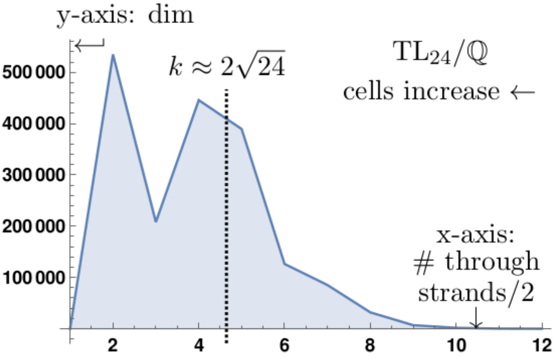
Vorgelegt von H. Weyl in der Sitzung am 28 Oktober 1932.





It has been rediscovered many times

Rep gap and monoidal categories



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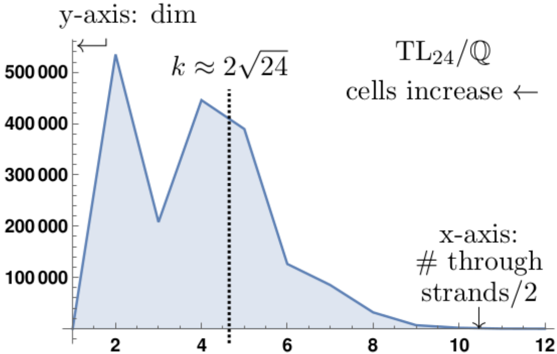
- Fact There is one simple TL_n-rep for each through strand $i \in \{n, n-2, ...,\}$
- Fact The simple dims are known recursively, see e.g. **Andersen ∼2017, Spencer ∼2020**
- Fact The simple dims behave as above, see e.g. **A computer ∼2021**

Rep gap and monoidal categories



,

On can define a truncation TL_n^k to get rid of the small reps

- Fact There is one simple TL_n-rep for each through strand $i \in \{n, n-2, ...,\}$
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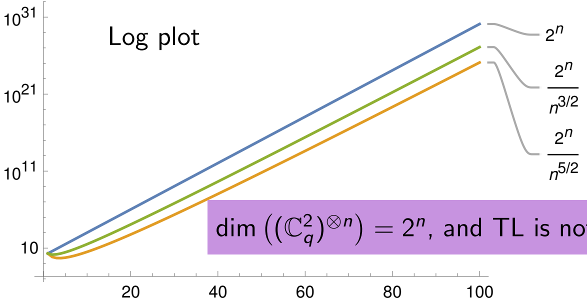
Rep gap and monoidal categories



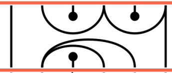
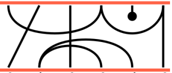
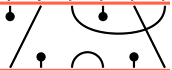
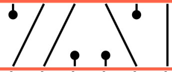
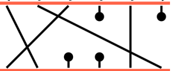
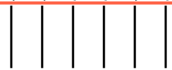
$\dim\left((\mathbb{C}_q^2)^{\otimes n}\right) = 2^n$, and TL is not far away!

Theorem For $0 \leq k \leq 2\sqrt{n}$ we have

$$rep_{\mathbb{Q}}(TL_n^k) \geq \frac{4}{(n+2\sqrt{n}+2)(n+2\sqrt{n}+4)}\binom{n}{\frac{n}{2}-\sqrt{n}} \in \Theta\left(n^{-5/2}\cdot 2^n\right)$$

$$faith_{\mathbb{Q}}(TL_n^k) \geq \frac{6}{n+4}\binom{n}{\lfloor \frac{n}{2}-1\rfloor} \in \Theta\left(n^{-3/2}\cdot 2^n\right)$$

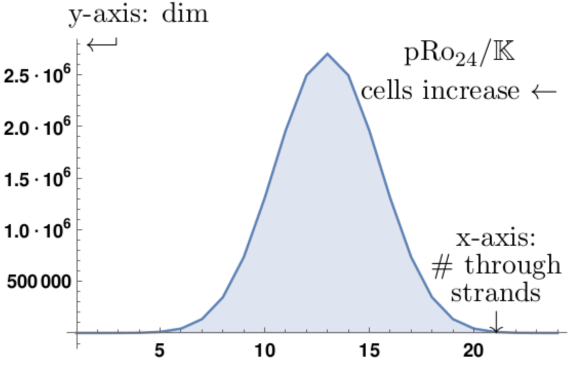
Rep gap and monoidal categories

Symbol	Diagrams	Symbol	Diagrams
pPa_n		Pa_n	
Mo_n		RoBr_n	
TL_n		Br_n	
pRo_n		Ro_n	
pS_n		S_n	

Summary

- Similar formulas hold for *gap* and *faith* but details are unknown
- The rep gap of monoids from monoidal categories is often large
- This is in particularly true for most of the “Schur–Weyl monoids” above

Rep gap and monoidal categories



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Summary

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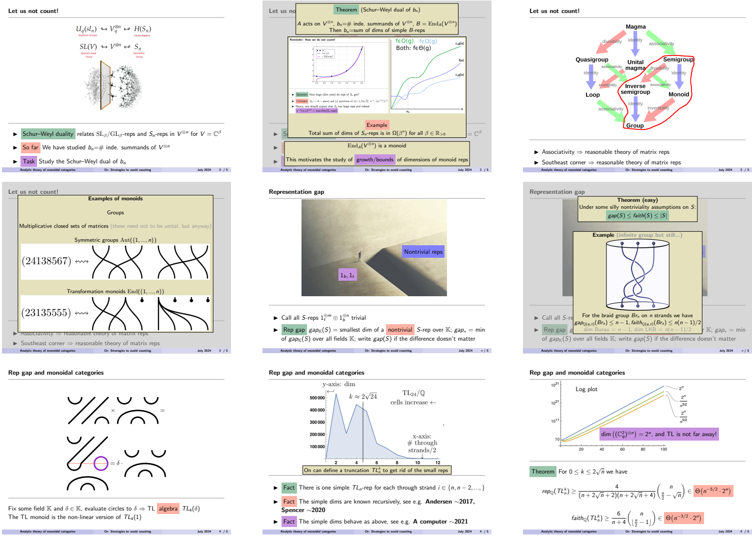
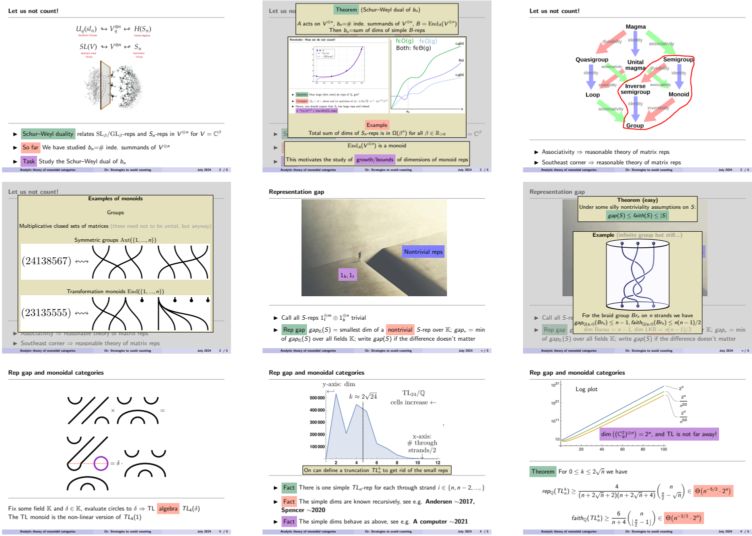


There is still much to do...



Thanks for your attention!