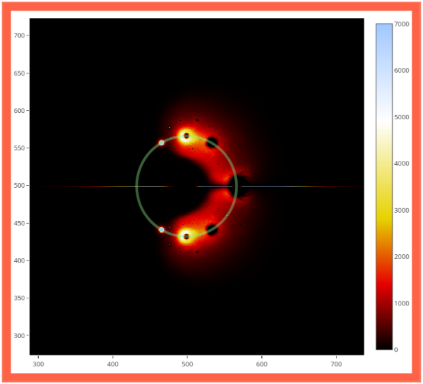
How good are (quantum) knot invariants?

Or: 1/4 of a century wasted!?

~~Accept~~Change what you cannot ~~change~~accept



I report on work of Dłotko–Gurnari–Sazdanovic + Zhang

Quantum invariants

CINQUIÈME COMPLÉMENT À L'ANALYSIS SITUS.

Par M. H. Poincaré, à Paris.

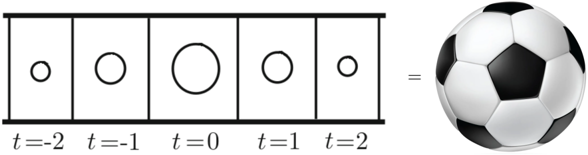
Adunanza del 22 novembre 1903.

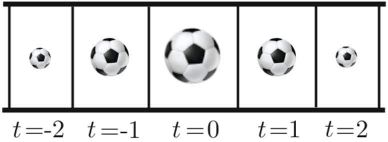
Il resterait une question à traiter :

Est-il possible que le groupe fondamental de V se réduise à la substitution identique, et que pourtant V ne soit pas simplement connexe ?

- Closed 3d manifolds need four-space to be realized, so are hard to imagine
- Poincaré ~1904 : classification in 3d is difficult, but maybe:
- Question The only closed simply connected 3d manifold is a sphere?

Quantum invariants

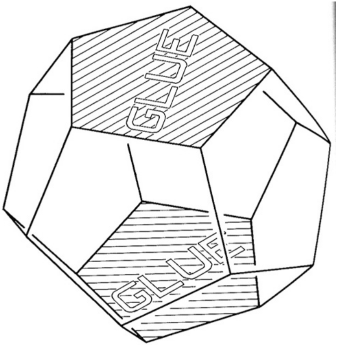


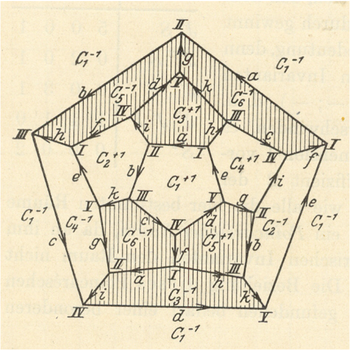


= 3*d* sphere S^3

- The answer to Poincaré's question is Yes! (Due to many people, finalized by Perelman ∼2002)
- The >3 dim analog was known for some time due to many people, e.g. Smale ∼1961 for >4 and Freedman ∼1982 for $=4$
- The smooth 4d version is "the last person standing in geometric topology"

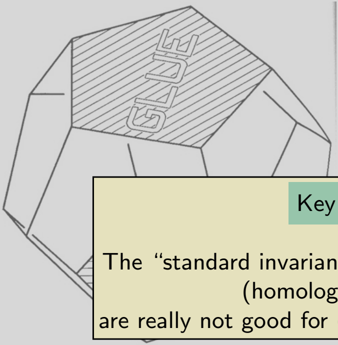
Quantum invariants

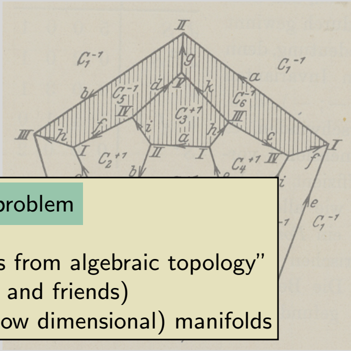




- The original "Poincaré conjecture" was homology detects the 3-sphere
- Poincaré found a counterexample ∼1904 (later reformulated as "gluing opposite sides of a dodecahedron") and then changed the "conjecture"
- Maybe this is why it was carefully called a question and not a conjecture

Quantum invariants





Key problem

The "standard invariants from algebraic topology"
(homology and friends)
are really not good for (low dimensional) manifolds

- The original "Poincaré conjecture" was homology detects the 3-sphere
- Poincaré found a counterexample ∼1904 (later reformulated as "gluing opposite sides of a dodecahedron") and then changed the "conjecture"
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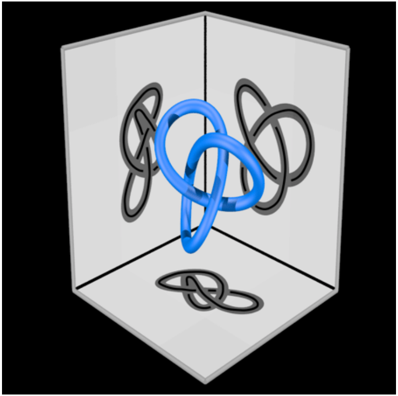
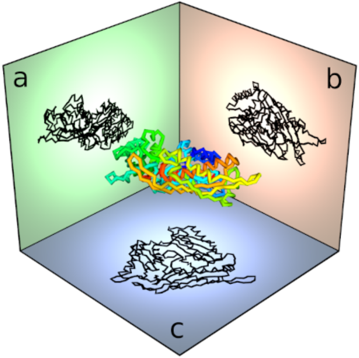
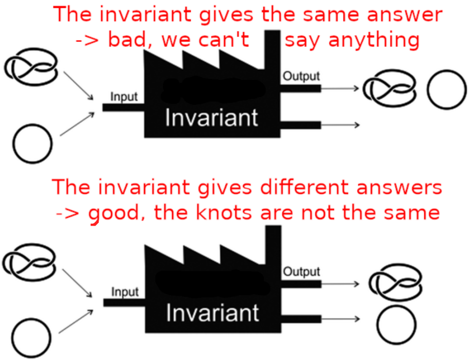
Quantum invariants





- Knot = closed string (a circle S^1) in three spaces; link = multiple components
- Knots are studied by projections to the plane Shadows
- Knots/links are the basic building blocks of low dimensional manifolds

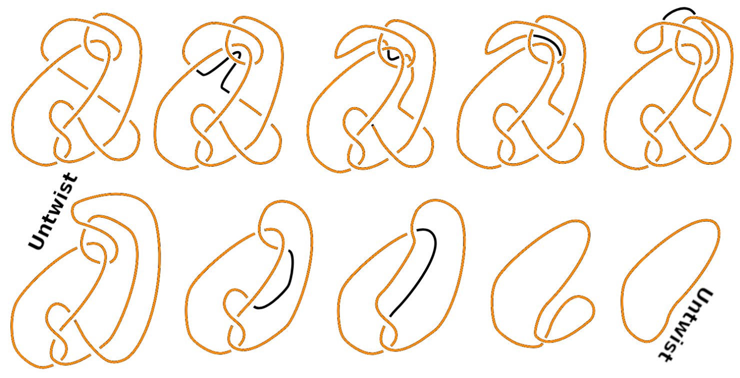
Quantum invariants



- In math knot theory started in the early 20th century
- Topologists from ∼1900-1980 studied knots from the point of view of invariants from homology theory
- Problem The invariants obtained are not particularly strong

Quantum invariants

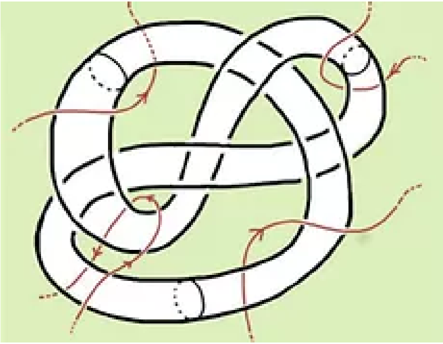
Even the unknotting problem is tricky



In general, knot theory was in need of new invariants
since the "standard invariants from algebraic topology"
(homology and friends)
are really not good for knots

- In
- To
  in
- Problem The invariants obtained are not particularly strong

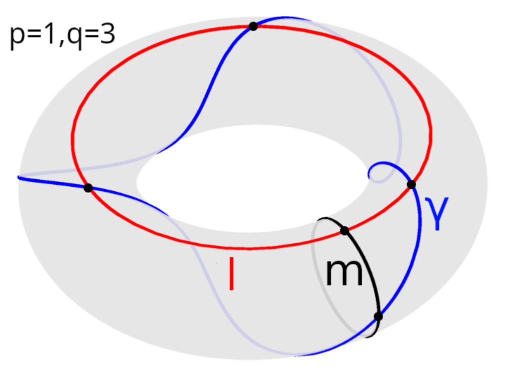
Quantum invariants



- A knot complement $S^3 \setminus \mathrm{int}(K)$ is a 3mfd bounding a torus
- Idea Glue back in a solid torus ST, but "twisted"
- Any such gluing is determined by the image of the meridian m, and m goes to some simple closed curve γ in $T = \partial ST$, and it hence suffices to describe γ

Quantum invariants

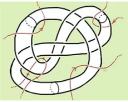
Write $[\gamma] = p \cdot [l] + q \cdot [m] \in H_1(\partial T)$



Surgery: We take out a torus T, fix γ determined by p, q and glue the meridian m of T back in on γ

- A knot c
- Idea Glu
- Any such gluing is determined by the image of the meridian m, and m goes to some simple closed curve γ in $T = \partial ST$, and it hence suffices to describe γ

Quantum invariants



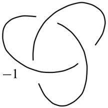
Every closed, orientable, connected 3mfd can be obtained by Dehn surgery, that is:

(i) Pick a finite collection of knots in S^3

(ii) Pick a surgery coefficient (p, q) for each knot

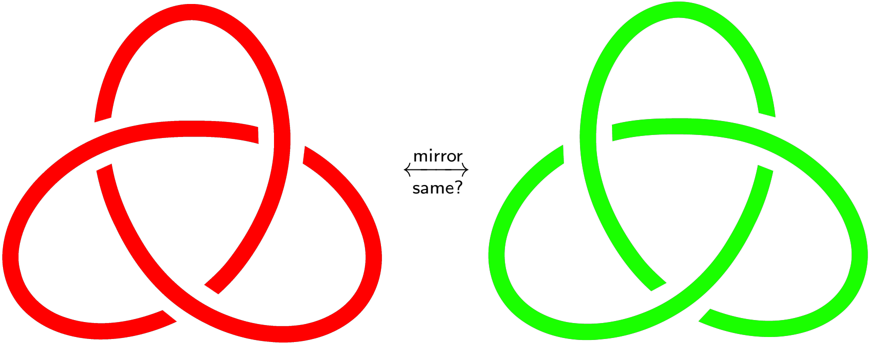
(iii) Perform the "remove-insert" surgery

- Every surgery on a knot gluing meridian to longitude gives a homology sphere



- New tools were needed!

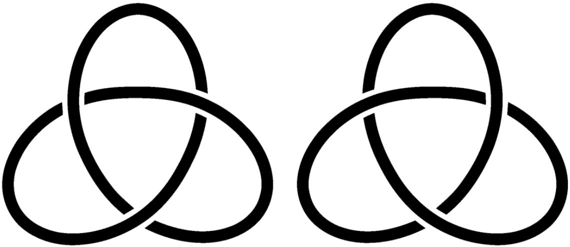
Quantum invariants



- Problem Deciding whether two knot projections are the same knot is difficult
- Task Find an invariant. Sounds easy? Well, most knot invariants are pretty bad...so: find a 'good' knot invariant
- Example There was no knot invariant that can distinguish the above knots

Jones' revolution (quantum invariants)

Left = right-handed trefoil? No!



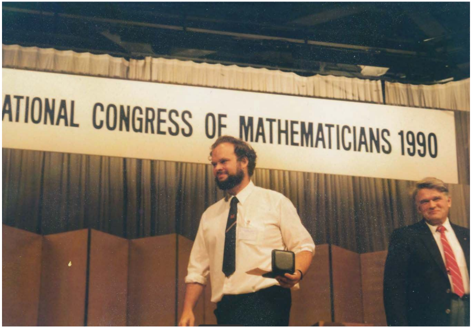
- The left-handed trefoil has Jones polynomial $-q^4 + q^3 + q$
- The right-handed trefoil has Jones polynomial $-q^{-4} + q^{-3} + q^{-1}$
- Thus, they are different

A zoo of quantum invariants For any semisimple Lie algebra and any representation:

Jones ∼1985 + friends There are polynomial knot/3mfd invariants

Khovanov ∼1999 + friends There are homological knot/3mfd/4mfd invariants

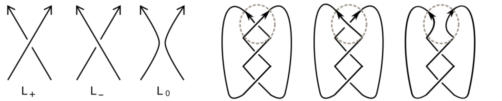
Quantum invariants



- Kyoto 1990 Jones receives the fields medal (with Faddeev in the background)
- Quote "Jones discovered an astonishing relationship between von Neumann algebras and geometric topology. As a result, they found a new polynomial invariant for knots and links in 3-space."
- Today The focus is on the quantum knot invariants à la Jones

Quantum invariants

Example (of quantum invariants)



- Alexander $(q^{1/2} - q^{-1/2}) \cdot \Delta_{L_0}(q) = \Delta_{L_+}(q) - \Delta_{L_-}(q)$
- Jones polynomial:
  - Skein relation $(q^{1/2} - q^{-1/2}) \cdot J_{L_0}(q) = q^{-1} \cdot J_{L_+}(q) - q \cdot J_{L_-}(q)$
  - Hecke algebra of the braid group
  - Quantum field theory as the unknot normalized vacuum expectation value of the Wilson loop operator in $SU(2)$ Chern–Simons gauge theory
- HOMFLY-PT: $z \cdot H_{L_0}(q) = a \cdot H_{L_+}(q) - a^{-1} \cdot H_{L_-}(q)$
- Khovanov homology – categorification of the Jones polynomial

Everyone loves them (I have spend **1/4 of a century** studying them) and they triggered a lot of research in

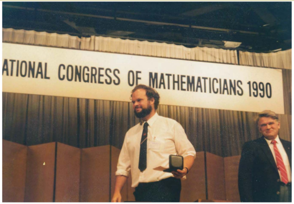
low dim topology, mathematical physics, quantum computing, ...

Question How good are these invariants (say, on prime knots)?

- Ky… …kground)
- Qu… …eumann alge… …nomial inva…
- Today The focus is on the quantum knot invariants à la Jones

Quantum invariants

They are loved because they relate many fields



- Kyoto 1990 Jones receives the fields medal (with Faddeev in the background)
- Quote “Jones discovered an astonishing relationship between von Neumann algebras and geometric topology. As a result, they found a new polynomial invariant for knots and links in 3-space.”

But somehow, nobody (at least not me) ever checked how they actually perform!

invariant for knots and links in 3-space.”

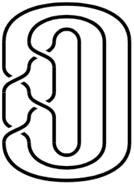
- Today The focus is on the quantum knot invariants à la Jones

Big data and knots

$R =$ , $R^{-1} =$ , $cap =$ , $cup =$ , $id = |$.

We associate these to linear maps (matrices upon choice of basis) denoted with the same symbols

(2D.1) $R, R^{-1} \colon V_q \otimes V_q \to V_q \otimes V_q, \quad cap \colon V_q \otimes V_q \to \mathbb{C}(q), \quad cup \colon \mathbb{C}(q) \to V_q \otimes V_q, \quad id \colon V_q \to V_q, v \mapsto v.$



$\leftrightsquigarrow$

$$0 \otimes cap \otimes 0$$
$$\circ$$
$$1 \otimes cap \otimes 1$$
$$\circ$$
$$2 \otimes cap \otimes 2$$
$$\circ$$
$$1 \otimes R^{-1} \otimes 3$$
$$\circ$$
$$0 \otimes R \otimes 4$$
$$\circ$$
$$1 \otimes R^{-1} \otimes 3$$
$$\circ$$
$$0 \otimes R \otimes 4$$
$$\circ$$
$$2 \otimes cup \otimes 2$$
$$\circ$$
$$1 \otimes cup \otimes 1$$
$$\circ$$
$$0 \otimes cup \otimes 0$$

- Construction of quantum invariants $(\mathfrak{g}, V_q)$ See above; here V_q is a representation of some semisimple Lie algebra $\mathfrak{g}$
- Black box Quantum groups give us the matrices
- Categorification There are also homology versions (defined similarly)

Big data and knots

Example

For the Jones polynomial J take $\mathfrak{g} = \mathfrak{sl}_2$, and $V_q = \mathbb{C}^2$

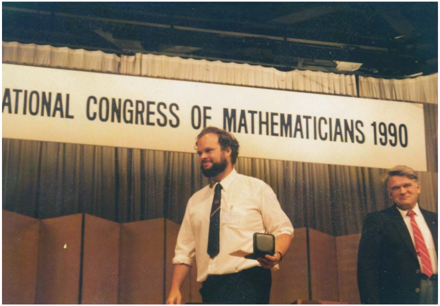
The R matrix is

$$R = \begin{pmatrix} q^{1/2} & 0 & 0 & 0 \\ 0 & 0 & q & 0 \\ 0 & q & q^{1/2} - q^{3/2} & 0 \\ 0 & 0 & 0 & q^{1/2} \end{pmatrix}$$

$q = 1$ gives the swap map

- Categorification There are also homology versions (defined similarly)

Big data and knots



(a) We start with the ***Jones polynomial*** or ***A1 invariant*** $(\mathfrak{sl}_2, \mathbb{C}^2, 0)$ (for the vector representation). This is our reference invariant.

(b) We investigate the ***2-colored Jones polynomial*** or ***B1 invariant*** $(\mathfrak{sl}_2, \mathrm{Sym}^2\mathbb{C}^2, 0)$ (for the simple three-dimensional representation). This is coloring.

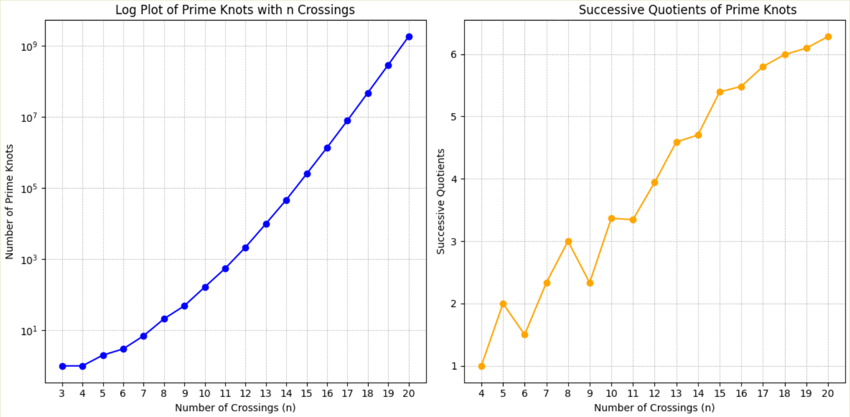
(c) We look at the ***A2 invariant*** $(\mathfrak{sl}_3, \mathbb{C}^3, 0)$ (for the vector representation). This is a rank increase.

(d) We then look at ***Khovanov homology*** or ***A1*** ^{c} ***invariant*** $(\mathfrak{sl}_2, \mathbb{C}^2, 1)$ (for the vector representation). This is categorification.

(e) Finally, we have the most classical knot polynomial, the ***Alexander polynomial*** or ***isotropic A1 invariant*** $(\mathfrak{gl}_{1|1}, \mathbb{C}^{1|1}, 0)$ (for the vector representation). Here we leave the realm of Lie algebras.

Big data and knots

Crucial



For this to work we need a lot of data; and we are lucky:

Ernst–Sumners $\sim$**1987** The number of knots grows exponential

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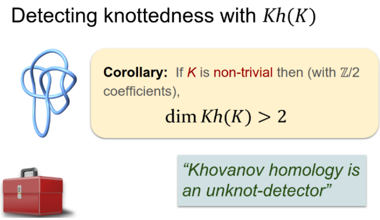
Big data and knots

Kronheimer–Mrowka gave a beautiful ICM talk about this (and related) breakthrough(s) Google 'Kronheimer Mrowka ICM 2018'



- **First measure** Put all (prime) knots in a bag, grab one randomly, how likely distinguishes, say, J the knot (from all others)?
- **More formally** What is

$$\lim_{n\to\infty} \#(\text{different } J \text{ with } \leq n \text{ crossings})/\#(\text{knots with } \leq n \text{ crossings})?$$

Big data and knots

Kro

brea

Small number coincidences?

KHOVANOV HOMOLOGY DETECTS:

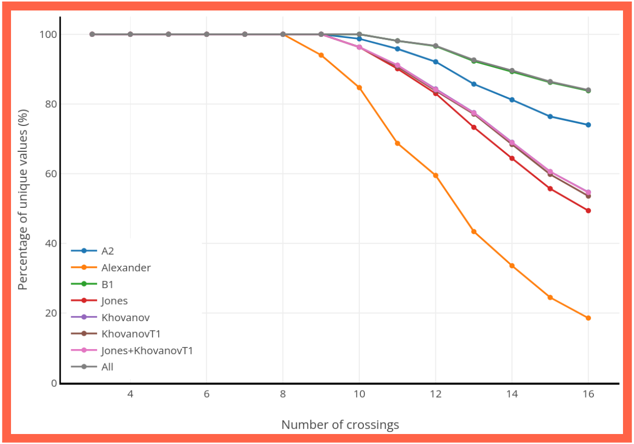
- The unknot: Kronheimer–Mrowka (2010)
- The unlink Hedden–Ni (2013), Batson–Seed (2015)
- The trefoils Baldwin–Sivek (2018)
- The Hopf link Baldwin–Sivek–Xie (2018)
- $2_1 \# 2_1$, the torus link $T(2,4)$ Xie–Zhang (2019)
- Split links Lipshitz–Sarkar (2019)
- The torus link $T(2,6)$ Martin (2020)
- $L6n1$ Xie–Zhang (2020)
- $L7n1$, $2_1 \# 3_1$ Li–Xie–Zhang (2020)
- Cinquefoil $T(5,2)$, non-fibered Baldwin, Siwek (2022)

▶ First meas ... omly, how likely distinguishe

▶ More formally What is

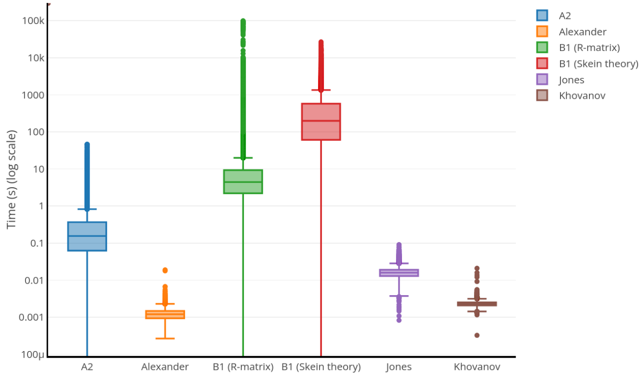
$$\lim_{n\to\infty} \#(\text{different } J \text{ with } \leq n \text{ crossings})/\#(\text{knots with } \leq n \text{ crossings})?$$

Big data and knots



- 1/4 century wasted!? They all distinguish knots with probability zero
- Data visualization gives us this conjecture and we can prove it for some of them

Even worse They all drop exponentially fast (proven in some cases)



If that is true, then the additional measure we would use is the computational complexity (in the number of crossings)

Invariant knot	A	A1	B1	J	K
Capital O	polynomial	$\approx 3^{\sqrt{n}}$	$\approx 3^{\sqrt{n}}$	$\approx 2^{\sqrt{n}}$	$\approx 2^{n}$ (maybe better)

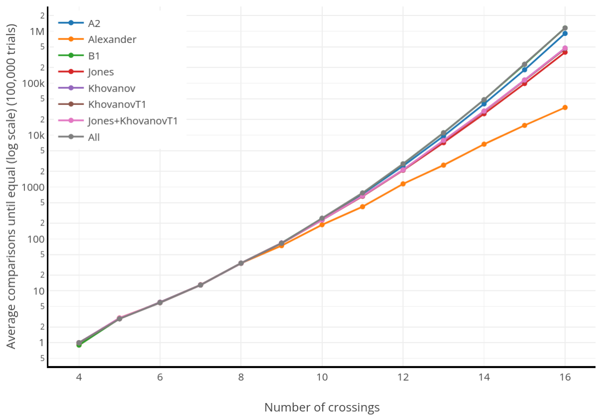
Alexander is then by far the best

Big data and

Some good news If we ask for measure 2:

Put all (prime) knots in a bag, grab two randomly how likely distinguishes, say, J these two?

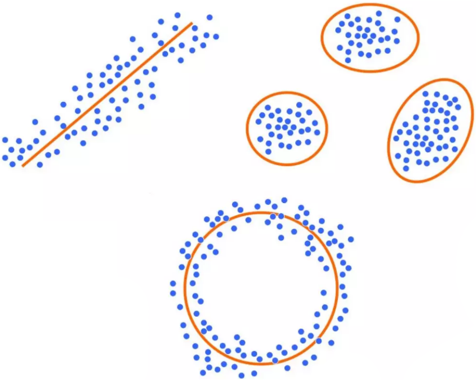
Data visualization gives us the conjecture that the probability is 1 (for all of them)



The complexity questions is however still lurking

- ▶ 1/4 cent… ty zero
- ▶ Data visu… some of them

Big data and knots - TDA



- TDA (topological data analysis) is the art of finding the shape of data
- Question What shape are quantum knot invariants?
- Question Can the shape measure how good they are?

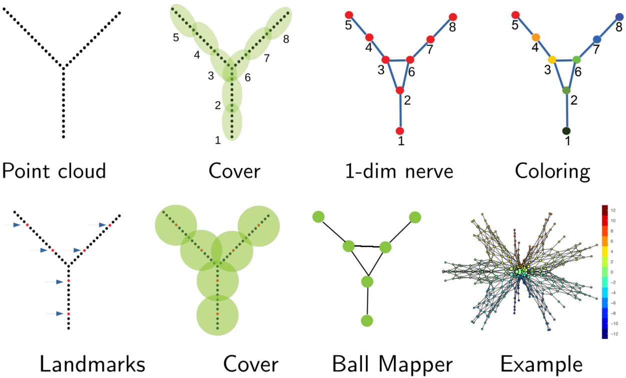
Knots form point clouds!

	q^{-3}	q^{-2}	q^{-1}	q^{0}	q^{1}	q^{2}	q^{3}	q^{4}	q^{5}	q^{6}	q^{7}
$J(0_1)$	0	0	0	1	0	0	0	0	0	0	0
$J(\mathrm{mir}(3_1))$	0	0	0	0	1	0	1	-1	0	0	0
$J(4_1)$	0	1	-1	1	-1	1	0	0	0	0	0
$J(\mathrm{mir}(5_1))$	0	0	0	0	0	1	0	1	-1	1	-1
$J(\mathrm{mir}(5_2))$	0	0	0	0	1	-1	2	-1	1	-1	0
$J(\mathrm{mir}(6_1))$	0	1	-1	2	-2	1	-1	1	0	0	0
$J(\mathrm{mir}(6_2))$	0	0	1	-1	2	-2	2	-2	1	0	0
$J(6_3)$	-1	2	-2	3	-2	2	-1	0	0	0	0

These are vectors in a 11d space

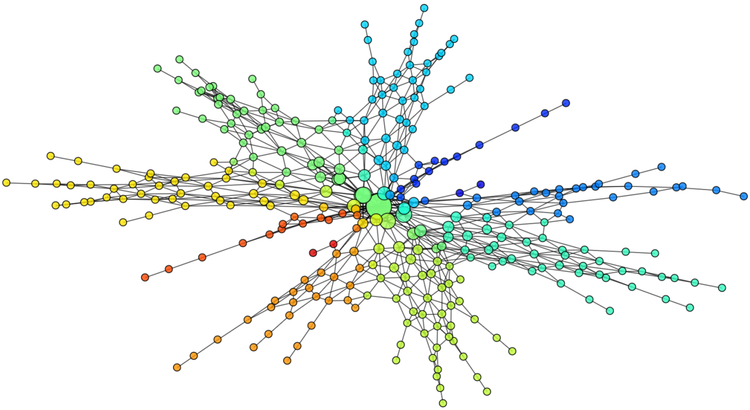
- Question Can the shape measure how good they are?

Big data and knots - TDA



- (Ball) Mapper = a way to turn point clouds into a graph
- Coloring gives additional information
- We see this in examples momentarily

Big data and knots - TDA

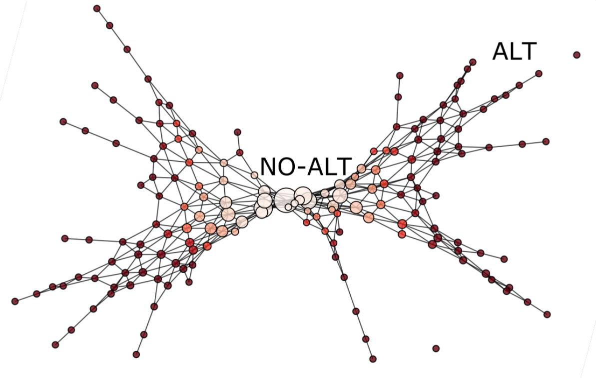


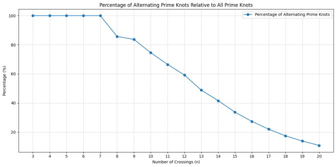
- Now live Ball mapper on knot data
- Play here https://dioscuri-tda.org/BallMapperKnots.html https://dustbringer.github.io/web–knot-invariant-comparison/

Data visualization

gives again many possible conjectures
and comparisons

An explanation why detecting alternating knots (but there a not many) is easy:

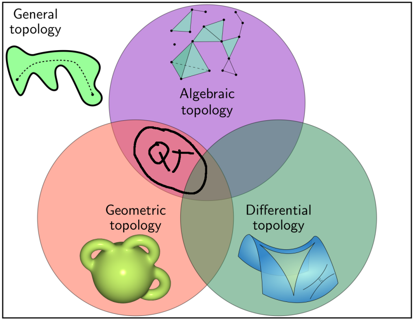




Most patterns that exists are probably to difficult to prove

https://dustbringer.github.io/web–knot-invariant-comparison/

Big data and knots - compare

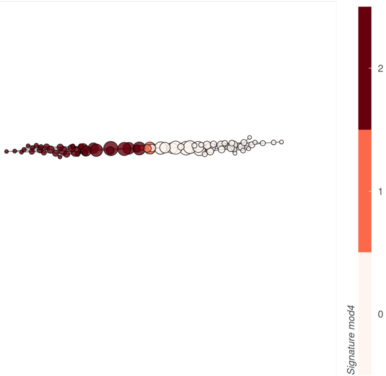


- Summary There is an infinite family of quantum invariants, “all” fail to detect knots fast and have superpolynomial runtime
- Essentially Before Jones we were missing knot invariants, after Jones we have too many and they are somewhat all them same
- Maybe what one should do instead is to compare them

Big data and knots - compare

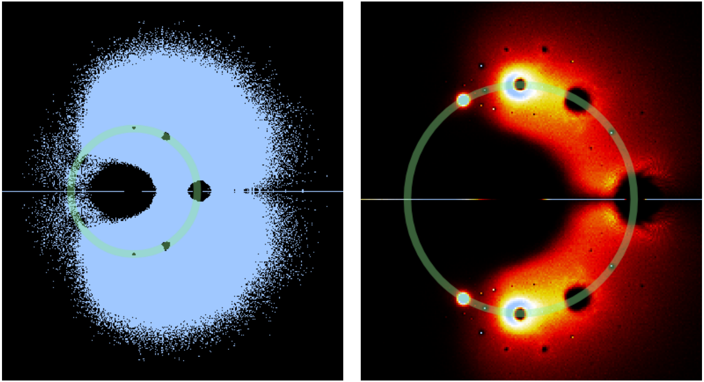
- Above Jones and its categorification (homology version)
- Categorification "=" pushing things further apart
- Comparing the invariants shows that they are related

Big data and knots - compare

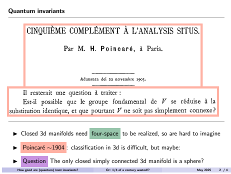


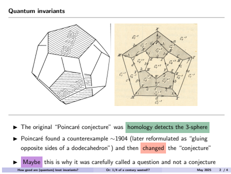
- Above Coloring of the Alexander invariant with the signature mode 4
- Signature = a traditional knot invariant (from homology)
- The eye catching conjecture is then easy to prove

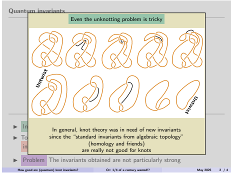
Big data and knots - compare

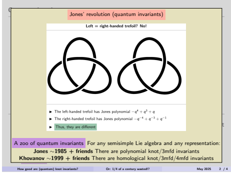


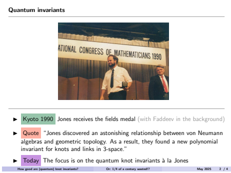
- Above The roots of the Jones polynomials
- This is a very specific distribution
- Another task Compare the distribution of the polynomials

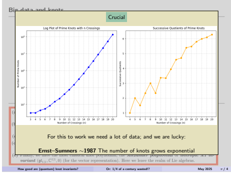


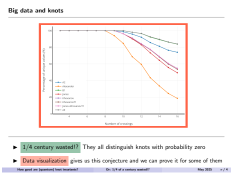


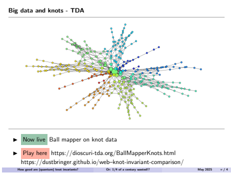


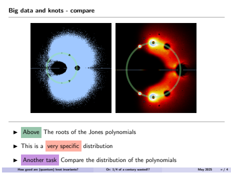




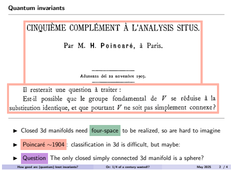


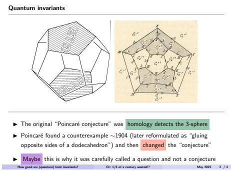


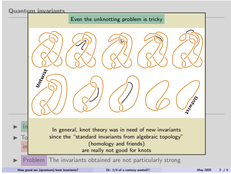


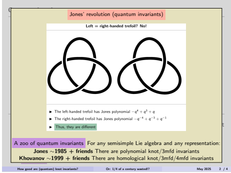


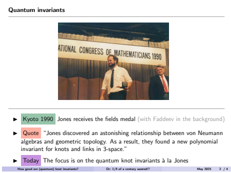
There is still much to do...

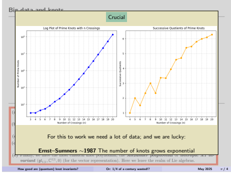


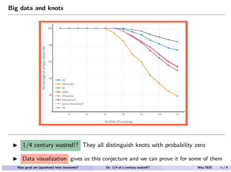


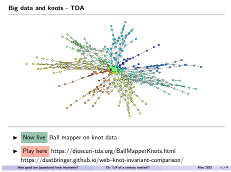


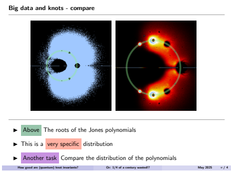












Thanks for your attention!