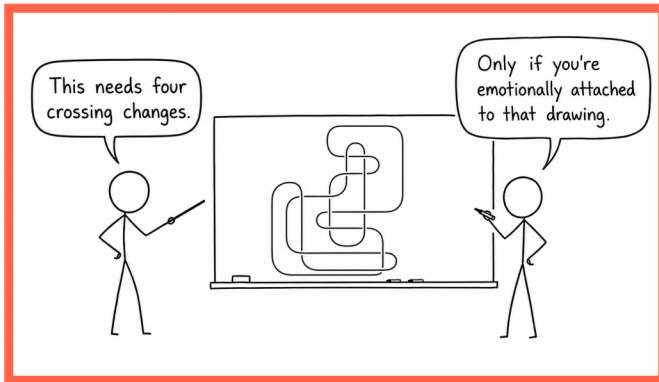


How many crossing changes?

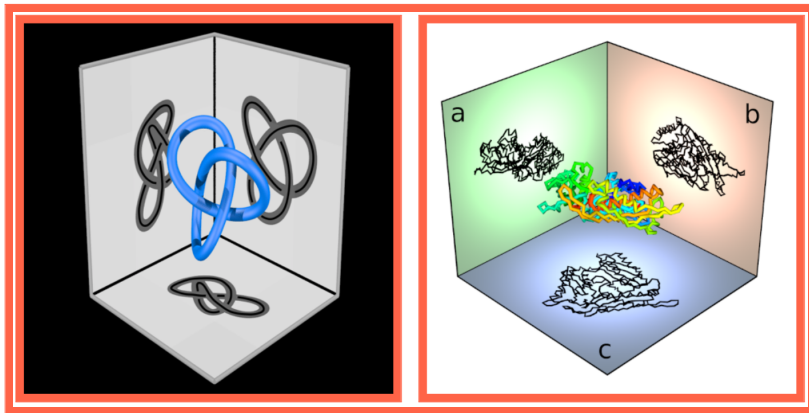
Or: Search all drawings, not one

Accept~~Change~~ what you cannot ~~change~~accept



I report on work of Anne Dranowski, Zhen Guo and Yura Kabkov, and many others

The unknotting number



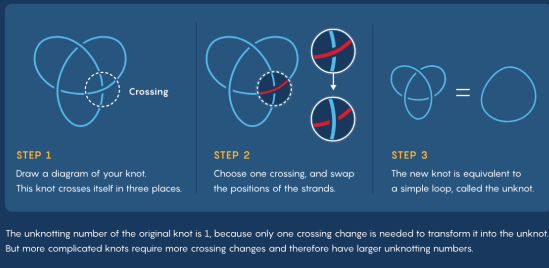
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- ▶ Knots are studied by projections, called diagrams, to the plane **Shadows**
- ▶ **Talk summary** Projecting is sometimes not a good idea

The unknotting number

$$\text{The unknotting number } u = u(K)$$

What Is an Unknotting Number?

To compare and measure the complexity of knots, mathematicians compute their unknotting numbers — the minimum number of steps it takes to transform a knot into a simple loop.



Mark Belen/Quanta Magazine

- Knot = The fewest crossing changes ("swaps") needed to unknot K .
- Knots are The minimum is over all diagrams of K .
- Talk summary That word "all" causes most of the trouble.
- Projecting is sometimes not a good idea

The unknotting number

AN INTERVAL IN A TABLE

$$u(K) \in [a, b]$$

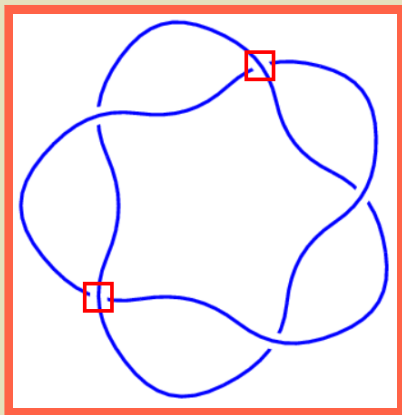
obstruction
proves $a \leq u$

explicit route
proves $u \leq b$

- Lower An obstruction rules out short routes
- Upper A concrete route proves that some number suffices
- Exact The number is known when the bounds meet

The unknotting number

Example



The displayed knot has $u = 2$.

The above crossing changes show $u \leq 2$ (swapping one over gives a trefoil).

The signature is -4 so $2 \leq u$ ($|sig|/2$ is a classical lower bound).

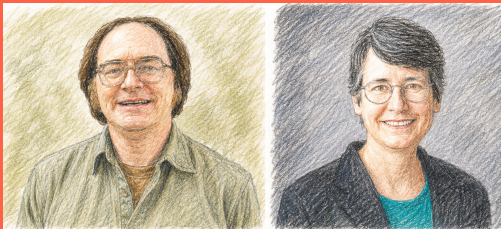
Exact The number is known when the bounds meet

The unknotting number



- **Tables** Knot tables record intervals $u(K) \in [a, b]$
- **Crazy** There are ten 10 crossing knots where u is not known, only $u \in [2, 3]$
- **Plan** Push the lower endpoint up and the upper one down

Why is that so difficult?

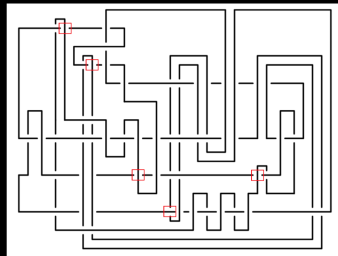


UNKNOTTING NUMBER IS NOT ADDITIVE UNDER CONNECTED SUM

MARK BRITTENHAM AND SUSAN HERMILLER

ABSTRACT. We give the first examples of a pair of knots K_1, K_2 in the 3-sphere for which their unknotting numbers satisfy $u(K_1 \# K_2) < u(K_1) + u(K_2)$. This answers question 1.69(B) from Kirby's problem list [24], in the negative.

Mathematics Subject Classification 2020: 57K10, 57K31



For the lower bound: Most invariants have nothing to say.

For the upper bound: We run over all (!) diagrams.

Above: A 56 crossing diagram of a 14 crossing knot showing $u \leq 5$.

This is very unexpected! u was conjectured to be six, and on all “small” diagrams one indeed needs 6 swaps.

Lower bounds: finding an obstruction

Signature of a knot

 Add languages

Article Talk

Read Edit View history

From Wikipedia, the free encyclopedia

The **signature of a knot** is a **topological invariant** in **knot theory**. It may be computed from the [Seifert surface](#).

Given a [knot](#) K in the [3-sphere](#), it has a [Seifert surface](#) S whose boundary is K . The **Seifert form** of S is the pairing $\phi : H_1(S) \times H_1(S) \rightarrow \mathbb{Z}$ given by taking the [linking number](#) $\text{lk}(a^+, b^-)$ where $a, b \in H_1(S)$ and a^+, b^- indicate the translates of a and b respectively in the positive and negative directions of the [normal bundle](#) to S .

Given a basis $b_1, \dots, b_{2g}$ for $H_1(S)$ (where g is the genus of the surface) the Seifert form can be represented as a $2g$ -by- $2g$ **Seifert matrix** V , $V_{ij} = \phi(b_i, b_j)$. The **signature** of the matrix $V + V^t$, thought of as a symmetric bilinear form, is the signature of the knot K .

[Slice knots](#) are known to have zero signature.

The Alexander module formulation [\[edit \]](#)

Knot signatures can also be defined in terms of the [Alexander module](#) of the knot complement. Let X be the universal abelian cover of the knot complement. Consider the Alexander module to be the first [homology group](#) of the universal abelian cover of the knot complement: $H_1(X; \mathbb{Q})$. Given a $\mathbb{Q}[\mathbb{Z}]$ -module V , let $\bar{V}$ denote the $\mathbb{Q}[\mathbb{Z}]$ -module whose underlying $\mathbb{Q}$ -module is V but where $\mathbb{Z}$ acts by the inverse covering transformation. Blanchfield's formulation of [Poincaré duality](#) for X gives a canonical isomorphism $H_1(X; \mathbb{Q}) \simeq \overline{H^2(X; \mathbb{Q})}$ where $H^2(X; \mathbb{Q})$ denotes the 2nd cohomology group of X with compact supports and coefficients in $\mathbb{Q}$. The universal coefficient theorem for $H^2(X; \mathbb{Q})$ gives a canonical isomorphism with $\text{Ext}_{\mathbb{Q}[\mathbb{Z}]}(H_1(X; \mathbb{Q}), \mathbb{Q}[\mathbb{Z}])$ (because the Alexander module is $\mathbb{Q}[\mathbb{Z}]$ -torsion). Moreover, just like in the [quadratic form formulation of Poincaré duality](#), there is a canonical isomorphism of $\mathbb{Q}[\mathbb{Z}]$ -modules $\text{Ext}_{\mathbb{Q}[\mathbb{Z}]}(H_1(X; \mathbb{Q}), \mathbb{Q}[\mathbb{Z}]) \simeq \text{Hom}_{\mathbb{Q}[\mathbb{Z}]}(H_1(X; \mathbb{Q}), [\mathbb{Q}[\mathbb{Z}]]/[\mathbb{Q}[\mathbb{Z}]])$, where $[\mathbb{Q}[\mathbb{Z}]]$ denotes the field of fractions of $\mathbb{Q}[\mathbb{Z}]$.



A Seifert surface bounded by a set ⁶⁷ of Borromean rings.

- ▶ **Classic** Signature gives an immediate obstruction: $|sig|/2 \leq u$
- ▶ **Good** It is easy to compute
- ▶ **Bad** It is very weak

Lower bounds: finding an obstruction

The first problem



Trefoil knot: tricolorable



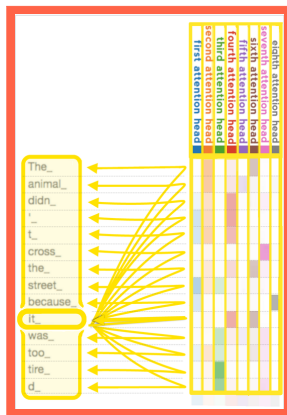
Figure-eight knot: NOT tricolorable

5W Infographics for Quanta Magazine

Most familiar invariants give little control over crossing swaps.
For example, the Jones polynomial
by itself gives little or no useful lower bound on $u(K)$.

It is very weak

Lower bounds: finding an obstruction



- ▶ Start with classical obstructions that react to crossing swaps
- ▶ Refine them, or turn them into finite calculations
- ▶ LLM's job Then run the resulting certificates over the database

Lower bounds: finding an obstruction

Breakfast for LLM's

ONE CERTIFICATE

$$K = 13n1457, \quad u(K) \in [1, 2]$$

$$\dim_{\mathbb{F}_4} \ker M_K(\omega) = 2$$



$$u(13n1457) = 2$$

Above: an example calculation.

Roughly, take a knot with $u \in [a, b]$, $a < b$.

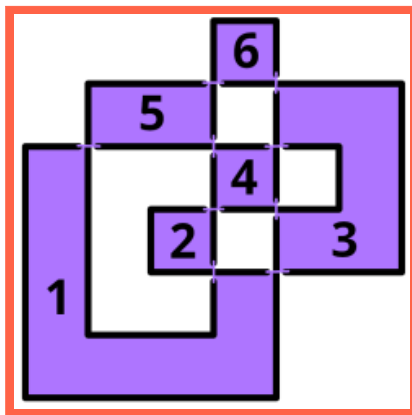
The new obstruction is then easy to compute from the database.

► Sta

► Ref

► LLM's job Then run the resulting certificates over the database

Lower bounds: finding an obstruction



- ▶ **Surface** A Seifert surface gives a matrix V
- ▶ **Module** $tV - V^T$ presents the Alexander module
- ▶ **Classic** Its generator number $e(K)$ satisfies $e(K) \leq u(K)$ (beefed-up signature)

Finite Alexander certificate

THE JULY 2026 SNAPSHOT

1787 exact values

273

2060 improved intervals

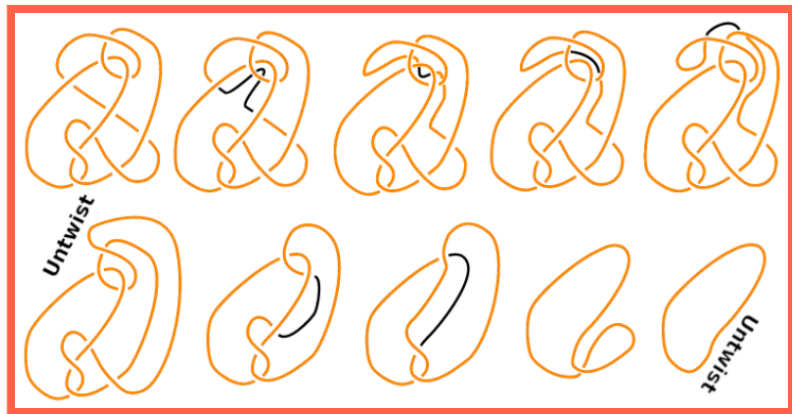
all counts deduplicated

Choose $\alpha \in \mathbb{F}^\times$ and compute the nullity of $\alpha V - V^T$.

Nullity r certifies $r \leq e(K) \leq u(K)$.

The calculation is finite, exact, and suitable for a database scan.

Upper bounds: Finding a route



- ▶ **Naive idea 1** Search over diagrams, not just crossings
- ▶ **Problem** The move graph is enormous (above: an unknot with a nasty reduction)
- ▶ **Need 1** A guide for which drawing to try next

Upper bounds: Finding a route

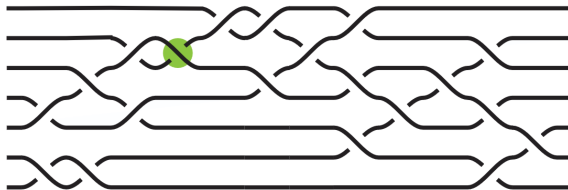
The catch

Lemma 2.5. *The knot $K13n3370$ has unknotting number less than or equal to 2.*

Proof. The knot $K13n3370$ is the closure of the 7-braid

$$\{1, 1, -3, 4, -3, -5, 5, -6, -6, 4, -5, 2, 4, -6, 3, 4, -1, 3, 5, 2\},$$

with 20 crossings, using the notational convention of Knotinfo, shown in Figure 3.



From <https://arxiv.org/abs/1705.05985> FIGURE 3. $K13n3370$

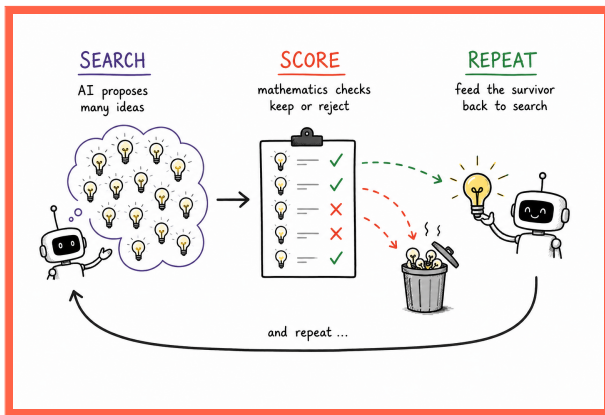
Exhaustive search grows too quickly.

Greedy simplification gets stuck.

A useful route may first add crossings.

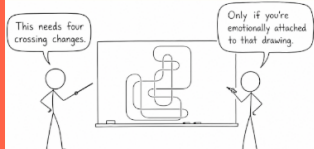
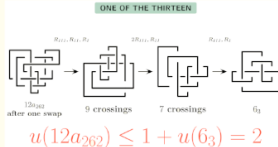
Example (above): $u \leq 2$ but all minimal diagrams need three swaps.

Upper bounds: Finding a route



- **State** One diagram of the knot
- **Move** Simplify, shuffle, or deliberately backtrack
- **Learn** Reward progress and learn what to try next

The pipeline


 $4_1 \# 9_{10}$


free moves

one swap

identify

standard
diagraminflated
diagramone-change
neighbourknown
 $u = 2$

$$u(4_1 \# 9_{10}) \leq 3$$

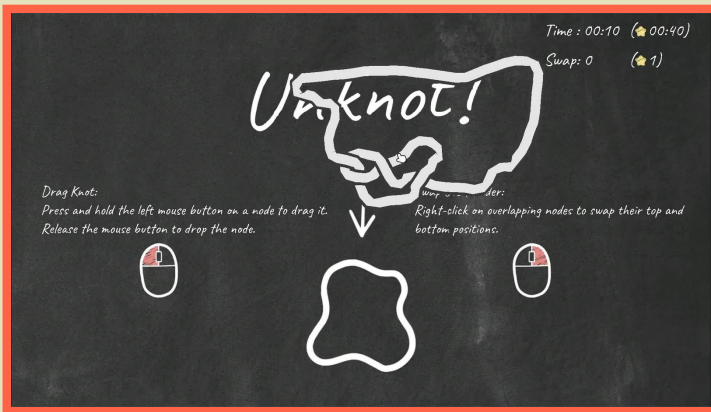
Try one swap, simplify, then identify.
 Current run: 8190 targets, 13 upper bounds improve.
 Five become exact; eight intervals tighten.

Upper bounds: Finding a route

	 HUMANS	AI 
<u>Superpower</u>	taste, intuition, weird ideas 	speed, scale, endless patience 
<u>What we bring</u>	a few shots worth taking 	thousands of shots 
<u>Not great at</u>	patience, repetition 	taste, knowing what to try 
<u>Best role</u>	choose directions, spot patterns 	explore everything, don't get bored 
<u>Result we want</u>	better guesses 	more (tested) candidates 

- ▶ Naive idea 2 Let humans do it, they have good intuition
- ▶ Problem They don't have any patience
- ▶ Need 2 A game that is fun (gamification of math problems)

The game (Unknot! available on Steam)



Drag to explore equivalent diagrams, and swap crossings.

Crucial: This is physics-based; no 2D projection.

Research levels: the ten 10 crossings knots with $u \in [2, 3]$.

Save and verify the play as an unknotting route.

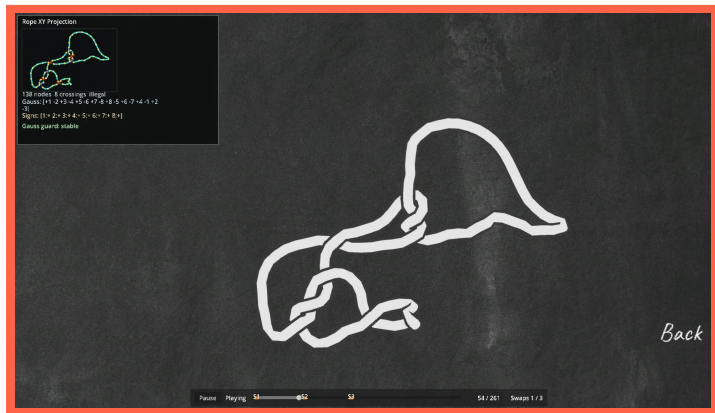
► Na

► Pro

► Need 2

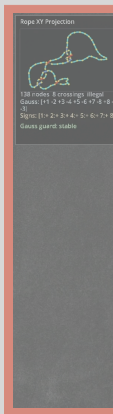
A game that is fun (gamification of math problems)

Upper bounds: Finding a route

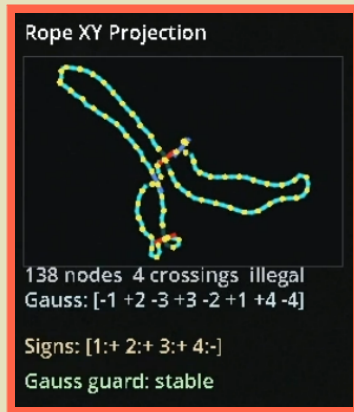


- ▶ **Replay** Save every move and every crossing swap
- ▶ **Rebuild** Recover a path through knot diagrams
- ▶ **Check** Turn the score into a proof

Upper bounds: Finding a route



From play to mathematics (the easy step)

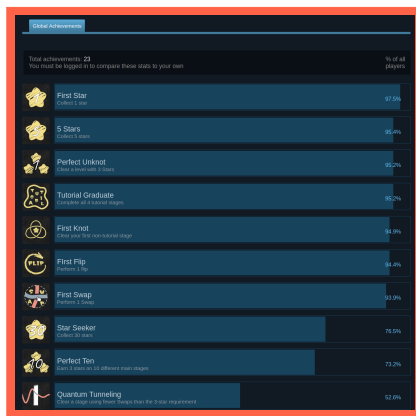


Back

- ▶ **Replay** Save
- ▶ **Rebuild** Rec
- ▶ **Check** Turn the score into a proof

Verify every isotopy and every crossing change.
Identify the final diagram independently.
Only then does the replay prove an upper bound.

Upper bounds: Finding a route



- ▶ Best case A player improves an upper bound
- ▶ Still useful We learn which diagrams humans explore
- ▶ The whole point Make math fun again! ;-) (Realistically: its an outreach project.)

Upper bounds: Finding a route

What this already did



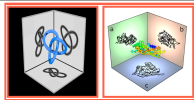
Unknot featured and various game shows (next: Tokyo Game Show).

Around 1000 players on Steam.

50ish tried research levels during game shows.

▶ The whole point Make math fun again! ;-) (Realistically: its an outreach project.)

The unknotting number



- ▶ **Knob** = closed string (a circle S^1) in three spaces; link = multiple components
- ▶ Knots are studied by projections, called diagrams, to the plane **Shadow**
- ▶ **Talk summary** Projecting is sometimes not a good idea

How many crossing changes? Or: Search all drawings, not one September 2026 5 / 5

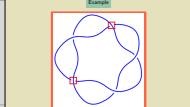
The unknotting number



- ▶ **Lower** An obstruction rules out short routes
- ▶ **Upper** A concrete route proves that some number suffices
- ▶ **Exact** The number is known when the bounds meet

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The unknotting number



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Why is that so difficult?



For the lower bound: Most invariants have nothing to say. For the upper bound: We just use all $\{7\}$ diagrams. Above: a 56 crossing diagram of a 14 crossing knot showing $u \leq 5$. This is very unexpected! u was conjectured to be $4n$, and an all "vertical" diagrams one indeed needs 5 swaps.

Upper bounds: Finding a route



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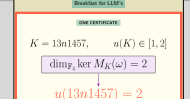
Lower bounds: finding an obstruction



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- ▶ **Good** Above: an example calculation. Roughly: take a knot with $u \in [a, b]$ $a < b$. The new obstruction is then easy to compute from the database.
- ▶ **LLM's job** Then run the resulting certificates over the database

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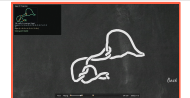
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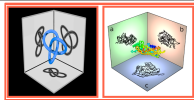


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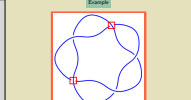
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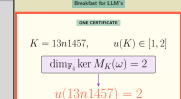
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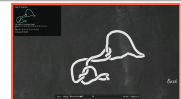
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Thanks for your attention!